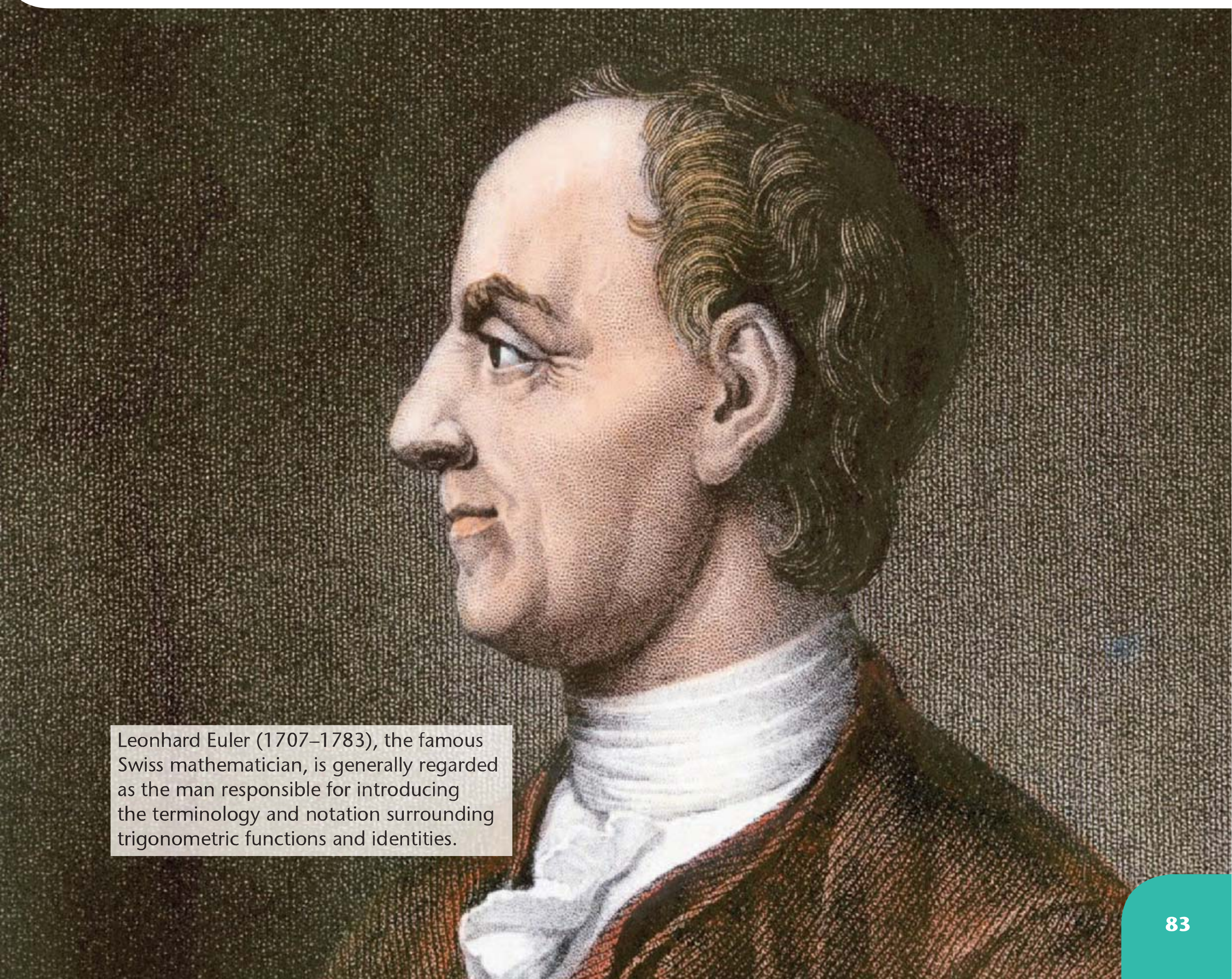
After completing this chapter you should know

1. the functions secant θ, cosecant θ and cotangent θ
2. the graphs of sec θ, cosec θ and cot θ
3. how to solve equations and prove identities involving sec θ, cosec θ and cot θ
4. how to prove and use the identities
   $1 + \tan^2\theta = \sec^2\theta$
   and $1 + \cot^2\theta = \text{cosec}^2\theta$
5. how to sketch and use the inverse trigonometric functions $\arcsin x$, $\arccos x$ and $\arctan x$.

6 Trigonometry



Leonhard Euler (1707–1783), the famous Swiss mathematician, is generally regarded as the man responsible for introducing the terminology and notation surrounding trigonometric functions and identities.

6.1 You need to know the functions secant θ, cosecant θ and cotangent θ.

■ **The functions secant θ, cosecant θ and cotangent θ are defined as:**

- $\sec\theta = \dfrac{1}{\cos\theta}$

  {undefined for values of θ at which $\cos\theta = 0$}

- $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

  {undefined for values of θ at which $\sin\theta = 0$}

- $\cot\theta = \dfrac{1}{\tan\theta}$

  {undefined for values of θ at which $\tan\theta = 0$}.

These are often written and pronounced as **sec θ**, **cosec θ** and **cot θ**.

Remember that $\cos^n\theta \equiv (\cos\theta)^n$ for $n \in \mathbb{Z}^+$. The convention is not used for $n \in \mathbb{Z}^-$.

For example, $\cos^{-1}\theta$ does not mean $\dfrac{1}{\cos\theta}$.

Do not confuse $\cos^{-1}\theta$ with $\sec\theta$.

As $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\cot\theta$ can also be written as $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

Example 1

Use your calculator to write down the value of:

a sec 280°

b cot 115°.

a $\sec 280° = \dfrac{1}{\cos 280°} = 5.76$ (3 s.f.)

Find cos 280° and then use the x^{-1} key.

b $\cot 115° = \dfrac{1}{\tan 115°} = -0.466$ (3 s.f.)

Find tan 115° and then use the x^{-1} key.

Example 2

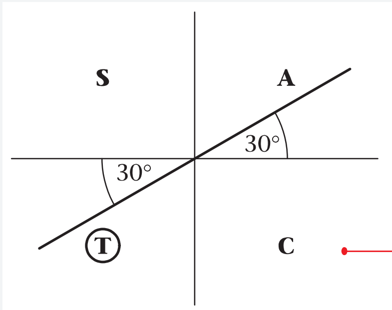
Work out the *exact* values of:

a sec 210°

b $\operatorname{cosec}\dfrac{3\pi}{4}$.

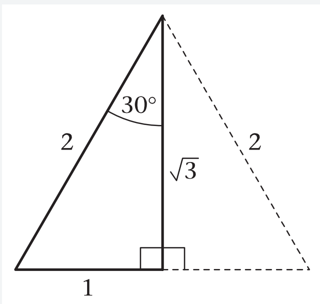
Exact here means give in surd form.

a $\sec 210° = \dfrac{1}{\cos 210°}$



210° is in 3rd quadrant, so $\cos 210° = -\cos 30°$.

So $\sec 210° = \dfrac{1}{-\cos 30°}$

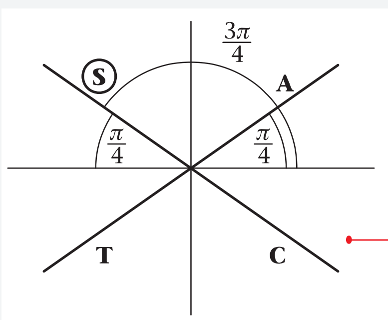


Remember that $\cos 30° = \dfrac{\sqrt{3}}{2}$, or draw an equilateral triangle of side 2 and use Pythagoras' theorem.

So $\sec 210° = -\dfrac{2}{\sqrt{3}}$ or $-\dfrac{2\sqrt{3}}{3}$

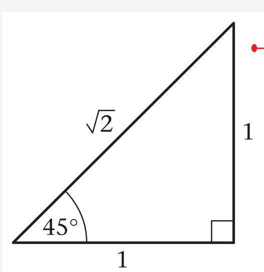
Rationalise the denominator.

b $\operatorname{cosec}\dfrac{3\pi}{4} = \dfrac{1}{\sin\left(\dfrac{3\pi}{4}\right)}$



$\dfrac{3\pi}{4}$ (135°) is in the 2nd quadrant, so $\sin\dfrac{3\pi}{4} = +\sin\dfrac{\pi}{4}$.

So $\operatorname{cosec}\left(\dfrac{3\pi}{4}\right) = \dfrac{1}{\sin\left(\dfrac{\pi}{4}\right)}$



Remember that $\sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$, or draw a right-angled isosceles triangle and use Pythagoras' theorem.

So $\operatorname{cosec}\dfrac{3\pi}{4} = \sqrt{2}$

Exercise 6A

1 Without using your calculator, write down the sign of the following trigonometric ratios:

a sec 300° **b** cosec 190° **c** cot 110°

d cot 200° **e** sec 95°

2 Use your calculator to find, to 3 significant figures, the values of

a sec 100° **b** cosec 260° **c** cosec 280°

d cot 550° **e** $\cot\frac{4\pi}{3}$ **f** $\sec 2.4^{c}$

g $\operatorname{cosec}\frac{11\pi}{10}$ **h** $\sec 6^{c}$

3 Find the exact value (in surd form where appropriate) of the following:

a cosec 90° **b** cot 135° **c** sec 180°

d sec 240° **e** cosec 300° **f** cot(−45°)

g sec 60° **h** cosec (−210°) **i** sec 225°

j $\cot\frac{4\pi}{3}$ **k** $\sec\frac{11\pi}{6}$ **l** $\operatorname{cosec}\left(-\frac{3\pi}{4}\right)$

4 **a** Copy and complete the table, showing values (to 2 decimal places) of sec θ for selected values of θ.

θ	0°	30°	45°	60°	70°	80°	85°	95°	100°	110°	120°	135°	150°	180°	210°
sec θ	1		1.41			5.76	11.47			−2.92		−1.41			−1.15

b Copy and complete the table, showing values (to 2 decimal places) of cosec θ for selected values of θ.

θ	10°	20°	30°	45°	60°	80°	90°	100°	120°	135°	150°	160°	170°
cosec θ				1.41			1		1.15	1.41			

θ	190°	200°	210°	225°	240°	270°	300°	315°	330°	340°	350°	390°
cosec θ					−1.15				−2			

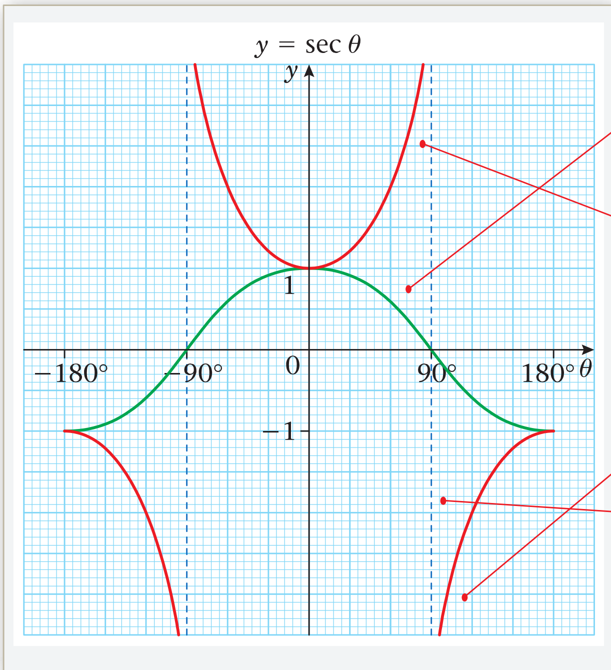
c Copy and complete the table, showing values (to 2 decimal places) of cot θ for selected values of θ.

θ	−90°	−60°	−45°	−30°	−10°	10°	30°	45°	60°	90°	120°	135°	150°	170°	210°	225°	240°	270°
cot θ	0	−0.58					1.73	1	0.58			−1					0.58	

6.2 You need to know the graphs of sec θ, cosec θ and cot θ.

Example 3

Sketch, in the interval $-180° \leqslant \theta \leqslant 180°$, the graph of $y = \sec\theta$.



First draw the graph $y = \cos\theta$.

For each value of θ, the value of $\sec\theta$ is the reciprocal of the corresponding value of $\cos\theta$.
In particular: as $\cos 0° = 1$, so $\sec 0° = 1$; as $\cos 180° = -1$, so $\sec 180° = -1$.

As θ approaches 90° from the left, $\cos\theta$ is +ve but approaches zero, and so $\sec\theta$ is +ve but becoming increasingly large.

As θ approaches 90° from the right, $\cos\theta$ is −ve but approaches zero, and so $\sec\theta$ is −ve but becoming increasingly large negative.

At $\theta = 90°$ there is no value of $\sec\theta$ (you may see $\pm\infty$ written for this value), so at $\theta = 90°$ there is a break in the curve; there is a vertical **asymptote** at this point.

Compare the completed table for Question 4a in Exercise 6A with the related part of the graph in Example 3.

- **The graph of $y = \sec\theta$, $\theta \in \mathbb{R}$, has symmetry in the y-axis and repeats itself every 360°. It has vertical asymptotes at all the values of θ for which $\cos\theta = 0$, i.e. at $\theta = 90° + 180n°$, $n \in \mathbb{Z}$.**

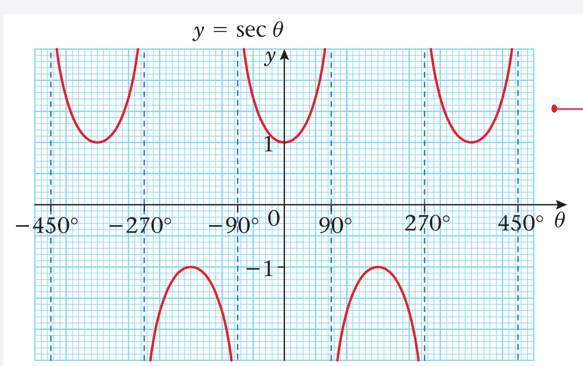
Example 4

Sketch the graph of $y = \operatorname{cosec}\theta$.

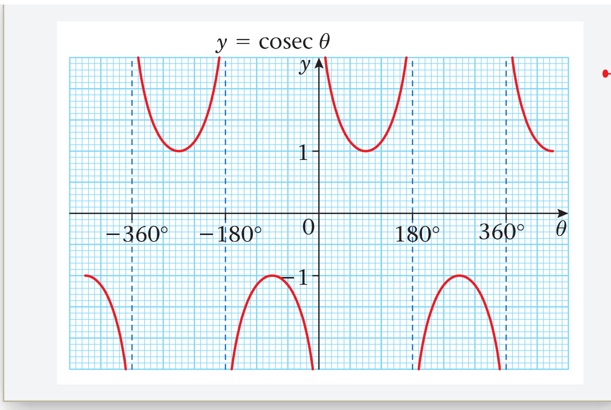
As $\sin\theta° = \cos(\theta - 90)°$,

it follows that $\operatorname{cosec}\theta° = \sec(\theta - 90)°$.

See Chapter 8 in Book C2.



First draw the graph of $y = \sec\theta$.



Then translate the graph of $y = \sec\theta$ by 90° to the right.

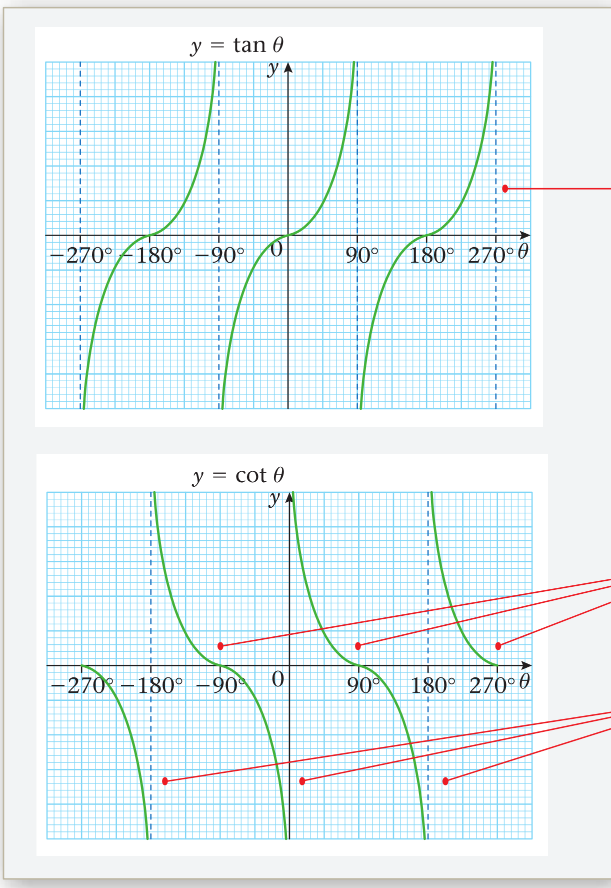
Note: You could first draw the graph of $y = \sin\theta$, and proceed as in Example 3.

Compare the completed table for Question 4b in Exercise 6A with the graph of $y = \operatorname{cosec}\theta$ in Example 4.

■ **The graph of $y = \operatorname{cosec}\theta$, $\theta \in \mathbb{R}$, has vertical asymptotes at all the values of θ for which $\sin\theta = 0$, i.e. at $\theta = 180n°$, $n \in \mathbb{Z}$, and the curve repeats itself every 360°.**

Example 5

Sketch the graph of $y = \cot\theta$.



First draw the graph $y = \tan\theta$.

At the values of θ where asymptotes occur on $y = \tan\theta$, the graph of $y = \cot\theta$ passes through the θ-axis.

At the values of θ where $y = \tan\theta$ crosses the θ-axis, $y = \cot\theta$ has asymptotes.

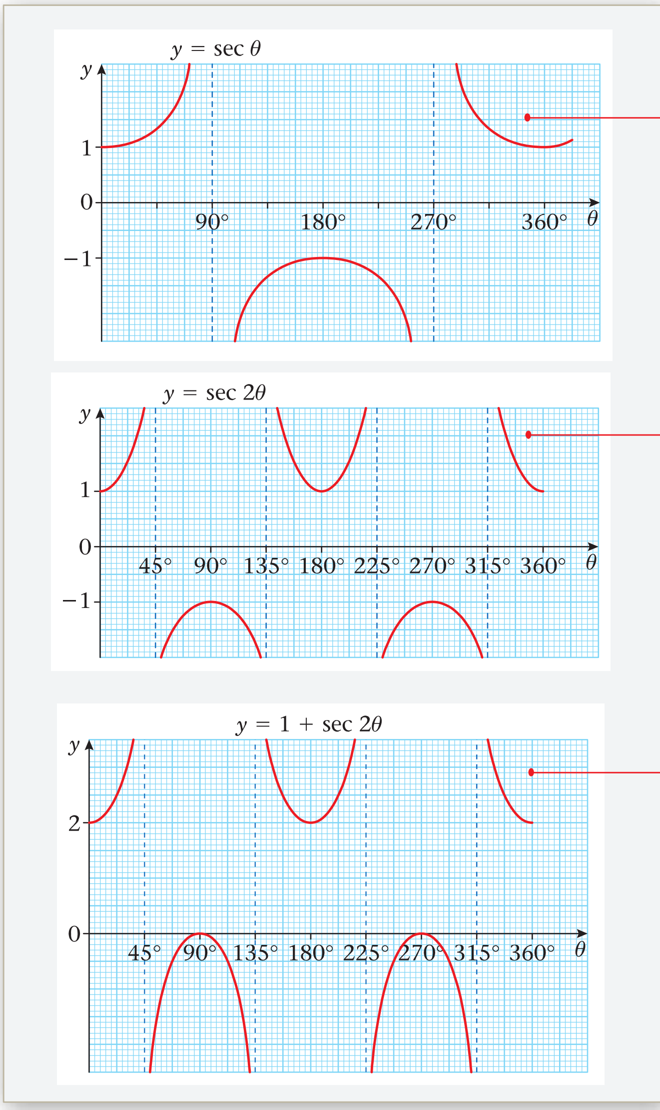
When $\tan\theta$ is small and positive, $\cot\theta$ is large and positive; when $\tan\theta$ is large and positive $\cot\theta$ is small and positive. Similarly for negative values.

Compare the graph in Example 5 with your answers to Exercise 6A, Question 4c.

■ **The graph of $y = \cot\theta$, $\theta \in \mathbb{R}$, has vertical asymptotes at all the values of θ for which $\sin\theta = 0$, i.e. at $\theta = 180n°$, $n \in \mathbb{Z}$, and the curve repeats itself every 180°.**

Example 6

Sketch, in the interval $0 \leqslant \theta \leqslant 360°$, the graph of $y = 1 + \sec 2\theta$.



Step 1
Draw the graph of $y = \sec \theta$.

Step 2
Stretch in the θ-direction with factor $\frac{1}{2}$.

Step 3
Translate by $+1$ in the y-direction.

Exercise 6B

1 **a** Sketch, in the interval $-540° \leqslant \theta \leqslant 540°$, the graphs of:
i $\sec \theta$ **ii** $\operatorname{cosec} \theta$ **iii** $\cot \theta$

b Write down the range of
i $\sec \theta$ **ii** $\operatorname{cosec} \theta$ **iii** $\cot \theta$

2 **a** Sketch, on the same set of axes, in the interval $0 \leqslant \theta \leqslant 360°$, the graphs of $y = \sec \theta$ and $y = -\cos \theta$.

b Explain how your graphs show that $\sec \theta = -\cos \theta$ has no solutions.

3 **a** Sketch, on the same set of axes, in the interval $0 \leqslant \theta \leqslant 360°$, the graphs of $y = \cot \theta$ and $y = \sin 2\theta$.

b Deduce the number of solutions of the equation $\cot \theta = \sin 2\theta$ in the interval $0 \leqslant \theta \leqslant 360°$.

4 **a** Sketch on separate axes, in the interval $0 \leqslant \theta \leqslant 360°$, the graphs of $y = \tan\theta$ and $y = \cot(\theta + 90°)$.

b Hence, state a relationship between $\tan\theta$ and $\cot(\theta + 90°)$.

5 **a** Describe the relationships between the graphs of

i $\tan\left(\theta + \frac{\pi}{2}\right)$ and $\tan\theta$ **ii** $\cot(-\theta)$ and $\cot\theta$

iii $\operatorname{cosec}\left(\theta + \frac{\pi}{4}\right)$ and $\operatorname{cosec}\theta$ **iv** $\sec\left(\theta - \frac{\pi}{4}\right)$ and $\sec\theta$

b By considering the graphs of $\tan\left(\theta + \frac{\pi}{2}\right)$, $\cot(-\theta)$, $\operatorname{cosec}\left(\theta + \frac{\pi}{4}\right)$ and $\sec\left(\theta - \frac{\pi}{4}\right)$, state which pairs of functions are equal.

6 Sketch on separate axes, in the interval $0 \leqslant \theta \leqslant 360°$, the graphs of:

a $y = \sec 2\theta$ **b** $y = -\operatorname{cosec}\theta$ **c** $y = 1 + \sec\theta$ **d** $y = \operatorname{cosec}(\theta - 30°)$

In each case show the coordinates of any maximum and minimum points, and of any points at which the curve meets the axes.

7 Write down the periods of the following functions. Give your answer in terms of π.

a $\sec 3\theta$ **b** $\operatorname{cosec}\frac{1}{2}\theta$ **c** $2\cot\theta$ **d** $\sec(-\theta)$

8 **a** Sketch the graph of $y = 1 + 2\sec\theta$ in the interval $-\pi \leqslant \theta \leqslant 2\pi$.

b Write down the y-coordinate of points at which the gradient is zero.

c Deduce the maximum and minimum values of $\dfrac{1}{1 + 2\sec\theta}$, and give the smallest positive values of θ at which they occur.

6.3 You need to be able to simplify expressions, prove identities and solve equations involving secant θ, cosecant θ and cotangent θ.

Example 7

Simplify

a $\sin\theta\cot\theta\sec\theta$ **b** $\sin\theta\cos\theta(\sec\theta + \operatorname{cosec}\theta)$

a $\sin\theta\cot\theta\sec\theta$

$= \cancel{\sin\theta}^{1} \times \dfrac{\cancel{\cos\theta}^{1}}{\cancel{\sin\theta}_{1}} \times \dfrac{1}{\cancel{\cos\theta}_{1}}$

Write the expression in terms of sin and cos, using $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ and $\sec\theta = \dfrac{1}{\cos\theta}$.

$= 1$

b $\sec\theta + \operatorname{cosec}\theta = \dfrac{1}{\cos\theta} + \dfrac{1}{\sin\theta}$

Write the expression in terms of sin and cos, using $\sec\theta = \dfrac{1}{\cos\theta}$ and $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$.

$= \dfrac{\sin\theta + \cos\theta}{\sin\theta\cos\theta}$

Put over common denominator.

So $\sin\theta\cos\theta(\sec\theta + \operatorname{cosec}\theta) = \sin\theta + \cos\theta$

Multiply both sides by $\sin\theta\cos\theta$.

The given expression reduces to $\sin\theta + \cos\theta$.

Example 8

Show that $\dfrac{\cot\theta\operatorname{cosec}\theta}{\sec^2\theta+\operatorname{cosec}^2\theta}\equiv\cos^3\theta$

Consider LHS:

The numerator $\cot\theta\operatorname{cosec}\theta$

$$\equiv\frac{\cos\theta}{\sin\theta}\times\frac{1}{\sin\theta}\equiv\frac{\cos\theta}{\sin^2\theta}$$

Write the expression in terms of sin and cos, using $\cot\theta=\dfrac{\cos\theta}{\sin\theta}$ and $\operatorname{cosec}\theta=\dfrac{1}{\sin\theta}$.

The denominator $\sec^2\theta+\operatorname{cosec}^2\theta$

$$\equiv\frac{1}{\cos^2\theta}+\frac{1}{\sin^2\theta}$$

$$\equiv\frac{\sin^2\theta+\cos^2\theta}{\cos^2\theta\sin^2\theta}$$

$$\equiv\frac{1}{\cos^2\theta\sin^2\theta}$$

Write the expression in terms of sin and cos, using $\sec^2\theta=\left(\dfrac{1}{\cos\theta}\right)^2\equiv\dfrac{1}{\cos^2\theta}$ and $\operatorname{cosec}^2\theta\equiv\dfrac{1}{\sin^2\theta}$.

Remember that $\sin^2\theta+\cos^2\theta\equiv1$.

So $\dfrac{\cot\theta\operatorname{cosec}\theta}{\sec^2\theta+\operatorname{cosec}^2\theta}$

$$\equiv\left(\frac{\cos\theta}{\sin^2\theta}\right)\div\left(\frac{1}{\cos^2\theta\sin^2\theta}\right)$$

$$\equiv\frac{\cos\theta}{\sin^2\theta}\times\frac{\cos^2\theta\sin^2\theta}{1}$$

$$\equiv\cos^3\theta$$

Remember to invert the fraction when changing from ÷ sign to ×.

Example 9

Solve the equations:

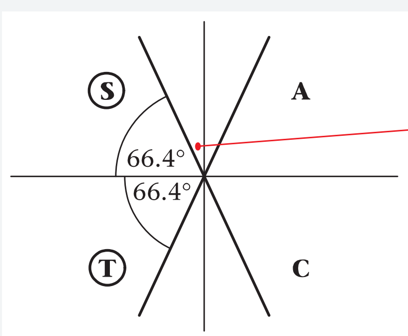
a $\sec\theta=-2.5$ **b** $\cot2\theta=0.6$

in the interval $0\leqslant\theta\leqslant360°$.

a As $\sec\theta=-2.5$

so $\cos\theta=-0.4$

Use $\cos\theta=\dfrac{1}{\sec\theta}$ to rewrite as $\cos\theta=\ldots$.



As $\cos\theta$ is −ve, θ is in 2nd and 3rd quadrants.

Remember that if you are using the quadrant diagram, the acute angle to the horizontal is $\cos^{-1}(+0.4)$.

$\theta=113.6°, 246.4°=114°, 246°$ (3 s.f.)

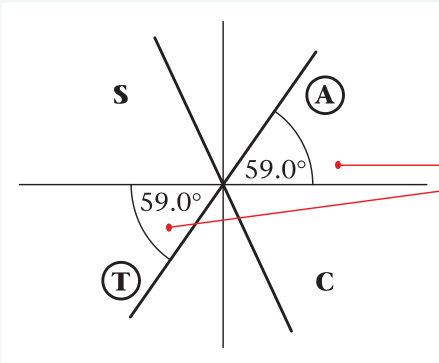
Read off from the diagram.

b As $\cot 2\theta = 0.6$

so $\tan 2\theta = \frac{5}{3}$

Use $\tan 2\theta = \dfrac{1}{\cot 2\theta} = \dfrac{1}{\left(\frac{3}{5}\right)} = \frac{5}{3}$.

Let $X = 2\theta$, so that you are solving $\tan X = \frac{5}{3}$, in the interval $0 \leqslant X \leqslant 720°$.



Draw the quadrant diagram, with the acute angle $X = \tan^{-1}\frac{5}{3}$ drawn to the horizontal in the 1st and 3rd quadrants.

Remember that $X = 2\theta$.

$X = 59.0°, 239.0°, 419.0°, 599.0°$

So $\theta = 29.5°, 120°, 210°, 300°$ (3 s.f.)

Exercise 6C

Give solutions to these equations, correct to 1 decimal place.

1 Rewrite the following as powers of $\sec\theta$, $\operatorname{cosec}\theta$ or $\cot\theta$:

a $\dfrac{1}{\sin^3\theta}$ **b** $\sqrt{\dfrac{4}{\tan^6\theta}}$ **c** $\dfrac{1}{2\cos^2\theta}$ **d** $\dfrac{1-\sin^2\theta}{\sin^2\theta}$

e $\dfrac{\sec\theta}{\cos^4\theta}$ **f** $\sqrt{\operatorname{cosec}^3\theta\cot\theta\sec\theta}$ **g** $\dfrac{2}{\sqrt{\tan\theta}}$ **h** $\dfrac{\operatorname{cosec}^2\theta\tan^2\theta}{\cos\theta}$

2 Write down the value(s) of $\cot x$ in each of the following equations:

a $5\sin x = 4\cos x$ **b** $\tan x = -2$ **c** $3\dfrac{\sin x}{\cos x} = \dfrac{\cos x}{\sin x}$

3 Using the definitions of **sec**, **cosec**, **cot** and **tan** simplify the following expressions:

a $\sin\theta\cot\theta$

b $\tan\theta\cot\theta$

c $\tan 2\theta\operatorname{cosec} 2\theta$

d $\cos\theta\sin\theta(\cot\theta + \tan\theta)$

e $\sin^3 x\operatorname{cosec} x + \cos^3 x\sec x$

f $\sec A - \sec A\sin^2 A$

g $\sec^2 x\cos^5 x + \cot x\operatorname{cosec} x\sin^4 x$

4 Show that

a $\cos\theta + \sin\theta\tan\theta \equiv \sec\theta$

b $\cot\theta + \tan\theta \equiv \operatorname{cosec}\theta\sec\theta$

c $\operatorname{cosec}\theta - \sin\theta \equiv \cos\theta\cot\theta$

d $(1-\cos x)(1+\sec x) \equiv \sin x\tan x$

e $\dfrac{\cos x}{1-\sin x} + \dfrac{1-\sin x}{\cos x} \equiv 2\sec x$

f $\dfrac{\cos\theta}{1+\cot\theta} \equiv \dfrac{\sin\theta}{1+\tan\theta}$

5 Solve, for values of θ in the interval $0 \leqslant \theta \leqslant 360°$, the following equations. Give your answers to 3 significant figures where necessary.

a $\sec\theta = \sqrt{2}$ **b** $\operatorname{cosec}\theta = -3$ **c** $5\cot\theta = -2$ **d** $\operatorname{cosec}\theta = 2$

e $3\sec^2\theta - 4 = 0$ **f** $5\cos\theta = 3\cot\theta$ **g** $\cot^2\theta - 8\tan\theta = 0$ **h** $2\sin\theta = \operatorname{cosec}\theta$

6 Solve, for values of θ in the interval $-180° \leqslant \theta \leqslant 180°$, the following equations:

a $\operatorname{cosec}\theta = 1$ **b** $\sec\theta = -3$

c $\cot\theta = 3.45$ **d** $2\operatorname{cosec}^2\theta - 3\operatorname{cosec}\theta = 0$

e $\sec\theta = 2\cos\theta$ **f** $3\cot\theta = 2\sin\theta$

g $\operatorname{cosec} 2\theta = 4$ **h** $2\cot^2\theta - \cot\theta - 5 = 0$

7 Solve the following equations for values of θ in the interval $0 \leqslant \theta \leqslant 2\pi$. Give your answers in terms of π.

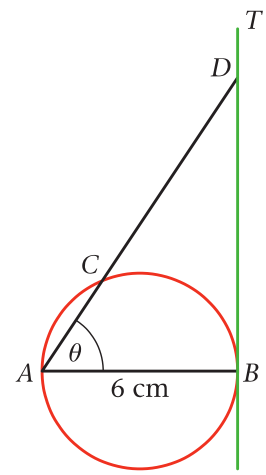
a $\sec\theta = -1$ **b** $\cot\theta = -\sqrt{3}$

c $\operatorname{cosec}\frac{1}{2}\theta = \dfrac{2\sqrt{3}}{3}$ **d** $\sec\theta = \sqrt{2}\tan\theta \left(\theta \neq \dfrac{\pi}{2}, \theta \neq \dfrac{3\pi}{2}\right)$

8 In the diagram $AB = 6$ cm is the diameter of the circle and BT is the tangent to the circle at B. The chord AC is extended to meet this tangent at D and $\angle DAB = \theta$.

a Show that $CD = 6(\sec\theta - \cos\theta)$.

b Given that $CD = 16$ cm, calculate the length of the chord AC.



6.4 **You need to know and be able to use the identities**

- $\mathbf{1 + \tan^2\theta \equiv \sec^2\theta}$
- $\mathbf{1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta}$

Example 10

Show that $1 + \tan^2\theta \equiv \sec^2\theta$

As $\sin^2\theta + \cos^2\theta \equiv 1$

so $\dfrac{\sin^2\theta}{\cos^2\theta} + \dfrac{\cos^2\theta}{\cos^2\theta} \equiv \dfrac{1}{\cos^2\theta}$

Divide both sides of the identity by $\cos^2\theta$.

so $\left(\dfrac{\sin\theta}{\cos\theta}\right)^2 + 1 \equiv \left(\dfrac{1}{\cos\theta}\right)^2$

Use $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\sec\theta = \dfrac{1}{\cos\theta}$

$\therefore$ $1 + \tan^2\theta \equiv \sec^2\theta$

Example 11

Show that $1 + \cot^2 \theta \equiv \operatorname{cosec}^2 \theta$

As $\quad \sin^2 \theta + \cos^2 \theta \equiv 1$

so $\quad \dfrac{\sin^2 \theta}{\sin^2 \theta} + \dfrac{\cos^2 \theta}{\sin^2 \theta} \equiv \dfrac{1}{\sin^2 \theta}$

Divide both sides of the identity by $\sin^2 \theta$.

so $\quad 1 + \left(\dfrac{\cos \theta}{\sin \theta}\right)^2 \equiv \left(\dfrac{1}{\sin \theta}\right)^2$

Use $\cot \theta = \dfrac{\cos \theta}{\sin \theta}$ and $\operatorname{cosec} \theta = \dfrac{1}{\sin \theta}$

$\therefore \quad 1 + \cot^2 \theta \equiv \operatorname{cosec}^2 \theta$

Example 12

Given that $\tan A = -\frac{5}{12}$, and that angle A is obtuse, find the exact value of

a $\sec A$ **b** $\sin A$

a Method 1

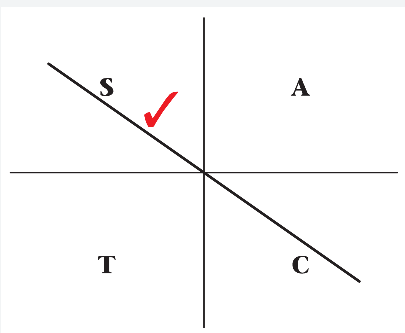
Using $1 + \tan^2 A \equiv \sec^2 A$

$\sec^2 A = 1 + \frac{25}{144} = \frac{169}{144}$

$\sec A = \pm\frac{13}{12}$

$\tan^2 A = \frac{25}{144}$

This does not take account of the fact that angle A is obtuse.

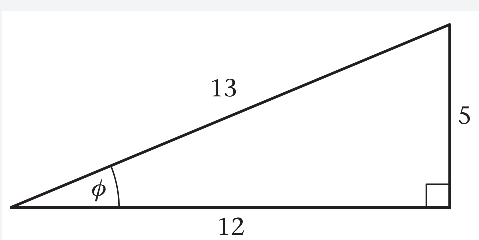


$\sec A = -\frac{13}{12}$

As angle A is obtuse, i.e. in the 2nd quadrant, sec A is −ve.

Method 2

Draw a right-angled triangle with $\tan \phi = \frac{5}{12}$.



Using Pythagoras' theorem, the hypotenuse is 13.

So $\quad \sec \phi = \frac{13}{12}$

Since $\cos \phi = \frac{12}{13}$

$\therefore \quad \sec A = -\frac{13}{12}$

Angle ϕ, in the 1st quadrant, is equally inclined to the horizontal as angle A, in the 2nd quadrant, and so all trigonometrical ratios of A are numerically equal to those of ϕ.

As A is in the 2nd quadrant, cos A is −ve and therefore sec A is −ve.

b Using $\tan A \equiv \dfrac{\sin A}{\cos A}$

$\sin A \equiv \tan A \cos A$

So $\sin A \equiv \left(-\frac{5}{12}\right) \times \left(-\frac{12}{13}\right)$

$\equiv \frac{5}{13}$

$\cos A = -\frac{12}{13}$, since $\cos A = \dfrac{1}{\sec A}$

Example 13

Prove the identities

a $\operatorname{cosec}^4 \theta - \cot^4 \theta \equiv \dfrac{1 + \cos^2 \theta}{1 - \cos^2 \theta}$

b $\sec^2 \theta - \cos^2 \theta \equiv \sin^2 \theta(1 + \sec^2 \theta)$

a LHS $= \operatorname{cosec}^4 \theta - \cot^4 \theta$

This is the difference of two squares, so factorise.

$\equiv (\operatorname{cosec}^2 \theta + \cot^2 \theta)(\operatorname{cosec}^2 \theta - \cot^2 \theta)$

$\equiv \operatorname{cosec}^2 \theta + \cot^2 \theta$

As $1 + \cot^2 \theta \equiv \operatorname{cosec}^2 \theta$, so $\operatorname{cosec}^2 \theta - \cot^2 \theta \equiv 1$.

$\equiv \dfrac{1}{\sin^2 \theta} + \dfrac{\cos^2 \theta}{\sin^2 \theta}$

Using $\operatorname{cosec} \theta \equiv \dfrac{1}{\sin \theta}$, $\cot \theta \equiv \dfrac{\cos \theta}{\sin \theta}$.

$\equiv \dfrac{1 + \cos^2 \theta}{\sin^2 \theta}$

$\equiv \dfrac{1 + \cos^2 \theta}{1 - \cos^2 \theta} =$ RHS

Using $\sin^2 \theta + \cos^2 \theta \equiv 1$.

b RHS $= \sin^2 \theta + \sin^2 \theta \sec^2 \theta$

Write in terms of $\sin \theta$ and $\cos \theta$.

Use $\sec \theta = \dfrac{1}{\cos \theta}$

$\equiv \sin^2 \theta + \dfrac{\sin^2 \theta}{\cos^2 \theta}$

$\equiv \sin^2 \theta + \tan^2 \theta$

$\dfrac{\sin^2 \theta}{\cos^2 \theta} = \left(\dfrac{\sin \theta}{\cos \theta}\right)^2 = \tan^2 \theta$.

$\equiv (1 - \cos^2 \theta) + (\sec^2 \theta - 1)$

Look at LHS. It is in terms of $\cos^2 \theta$ and $\sec^2 \theta$, so use $\sin^2 \theta + \cos^2 \theta \equiv 1$ and $1 + \tan^2 \theta \equiv \sec^2 \theta$.

Note: Try starting with the LHS, using $\cos^2 \theta \equiv 1 - \sin^2 \theta$ and $\sec^2 \theta \equiv 1 + \tan^2 \theta$.

$\equiv \sec^2 \theta - \cos^2 \theta$

$=$ LHS

The identities $1 + \tan^2 \theta \equiv \sec^2 \theta$ and $1 + \cot^2 \theta \equiv \operatorname{cosec}^2 \theta$ extend the range of equations that can be solved.

Example 14

Solve the equation $4\,\text{cosec}^2\,\theta - 9 = \cot\theta$, in the interval $0 \leqslant \theta \leqslant 360°$.

The equation can be rewritten as

$$4(1 + \cot^2\theta) - 9 = \cot\theta$$

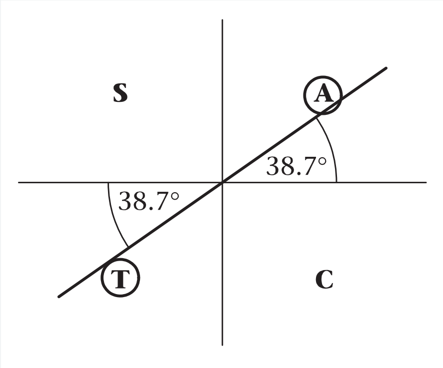
So $\quad 4\cot^2\theta - \cot\theta - 5 = 0$

$$(4\cot\theta - 5)(\cot\theta + 1) = 0$$

So $\quad \cot\theta = +\frac{5}{4}$ or $\cot\theta = -1$

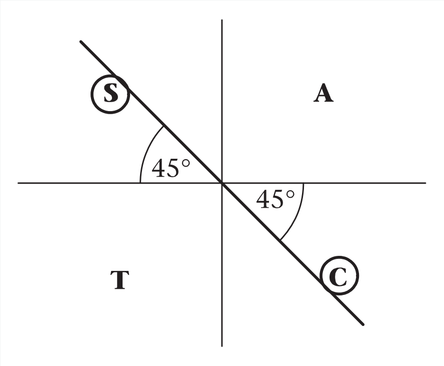
$\therefore \quad \tan\theta = +\frac{4}{5}$ or $\tan\theta = -1$

For $\tan\theta = +\frac{4}{5}$



$\theta = 38.7°, 219°$ (3 s.f.)

For $\tan\theta = -1$



$\theta = 135°, 315°$

This is a quadratic equation. You need to write it in terms of one trigonometrical function only, so use $1 + \cot^2\theta \equiv \text{cosec}^2\,\theta$.

Multiply out and re-order.

Factorise. You could use the quadratic formula.

As $\tan\theta$ is +ve, θ is in the 1st and 3rd quadrants. The acute angle to the horizontal is $\tan^{-1}\frac{4}{5} = 38.7°$.

Note: If α is the value the calculator gives for $\tan^{-1}\frac{4}{5}$, then the solutions are α and $(180° + \alpha)$.

As $\tan\theta$ is −ve, θ is in the 2nd and 4th quadrants. The acute angle to the horizontal is $\tan^{-1}1 = 45°$.

Note: If α is the value the calculator gives for $\tan^{-1}-1$ ($= -45°$), then the solutions are $(180° + \alpha)$ and $(360° + \alpha)$, as α is not in the given interval.

Exercise 6D

Give answers to 3 significant figures where necessary.

1 Simplify each of the following expressions:

a $1 + \tan^2\frac{1}{2}\theta$ **b** $(\sec\theta - 1)(\sec\theta + 1)$ **c** $\tan^2\theta(\text{cosec}^2\,\theta - 1)$

d $(\sec^2\theta - 1)\cot\theta$ **e** $(\text{cosec}^2\,\theta - \cot^2\theta)^2$ **f** $2 - \tan^2\theta + \sec^2\theta$

g $\dfrac{\tan\theta\sec\theta}{1 + \tan^2\theta}$ **h** $(1 - \sin^2\theta)(1 + \tan^2\theta)$ **i** $\dfrac{\text{cosec}\,\theta\cot\theta}{1 + \cot^2\theta}$

j $(\sec^4\theta - 2\sec^2\theta\tan^2\theta + \tan^4\theta)$ **k** $4\,\text{cosec}^2\,2\theta + 4\,\text{cosec}^2\,2\theta\cot^2 2\theta$

2 Given that $\operatorname{cosec} x = \dfrac{k}{\operatorname{cosec} x}$, where $k > 1$, find, in terms of k, possible values of $\cot x$.

3 Given that $\cot \theta = -\sqrt{3}$, and that $90° < \theta < 180°$, find the exact value of

a $\sin \theta$ **b** $\cos \theta$

4 Given that $\tan \theta = \frac{3}{4}$, and that $180° < \theta < 270°$, find the exact value of

a $\sec \theta$ **b** $\cos \theta$ **c** $\sin \theta$

5 Given that $\cos \theta = \frac{24}{25}$, and that θ is a reflex angle, find the exact value of

a $\tan \theta$ **b** $\operatorname{cosec} \theta$

6 Prove the following identities:

a $\sec^4 \theta - \tan^4 \theta \equiv \sec^2 \theta + \tan^2 \theta$

b $\operatorname{cosec}^2 x - \sin^2 x \equiv \cot^2 x + \cos^2 x$

c $\sec^2 A(\cot^2 A - \cos^2 A) \equiv \cot^2 A$

d $1 - \cos^2 \theta \equiv (\sec^2 \theta - 1)(1 - \sin^2 \theta)$

e $\dfrac{1 - \tan^2 A}{1 + \tan^2 A} \equiv 1 - 2\sin^2 A$

f $\sec^2 \theta + \operatorname{cosec}^2 \theta \equiv \sec^2 \theta \operatorname{cosec}^2 \theta$

g $\operatorname{cosec} A \sec^2 A \equiv \operatorname{cosec} A + \tan A \sec A$

h $(\sec \theta - \sin \theta)(\sec \theta + \sin \theta) \equiv \tan^2 \theta + \cos^2 \theta$

7 Given that $3\tan^2 \theta + 4\sec^2 \theta = 5$, and that θ is obtuse, find the exact value of $\sin \theta$.

8 Solve the following equations in the given intervals:

a $\sec^2 \theta = 3\tan \theta$, $0 \leqslant \theta \leqslant 360°$

b $\tan^2 \theta - 2\sec \theta + 1 = 0$, $-\pi \leqslant \theta \leqslant \pi$

c $\operatorname{cosec}^2 \theta + 1 = 3\cot \theta$, $-180° \leqslant \theta \leqslant 180°$

d $\cot \theta = 1 - \operatorname{cosec}^2 \theta$, $0 \leqslant \theta \leqslant 2\pi$

e $3\sec \frac{1}{2}\theta = 2\tan^2 \frac{1}{2}\theta$, $0 \leqslant \theta \leqslant 360°$

f $(\sec \theta - \cos \theta)^2 = \tan \theta - \sin^2 \theta$, $0 \leqslant \theta \leqslant \pi$

g $\tan^2 2\theta = \sec 2\theta - 1$, $0 \leqslant \theta \leqslant 180°$

h $\sec^2 \theta - (1 + \sqrt{3})\tan \theta + \sqrt{3} = 1$, $0 \leqslant \theta \leqslant 2\pi$

9 Given that $\tan^2 k = 2\sec k$,

a find the value of $\sec k$

b deduce that $\cos k = \sqrt{2} - 1$

c hence solve, in the interval $0 \leqslant k \leqslant 360°$, $\tan^2 k = 2\sec k$, giving your answers to 1 decimal place.

10 Given that $a = 4\sec x$, $b = \cos x$ and $c = \cot x$,

a express b in terms of a

b show that $c^2 = \dfrac{16}{a^2 - 16}$

11 Given that $x = \sec \theta + \tan \theta$,

a show that $\dfrac{1}{x} = \sec \theta - \tan \theta$.

b Hence express $x^2 + \dfrac{1}{x^2} + 2$ in terms of θ, in its simplest form.

12 Given that $2\sec^2 \theta - \tan^2 \theta = p$ show that $\operatorname{cosec}^2 \theta = \dfrac{p-1}{p-2}$, $p \neq 2$.

6.5 You need to be able to use the inverse trigonometric functions, arcsin x, arccos x and arctan x and their graphs.

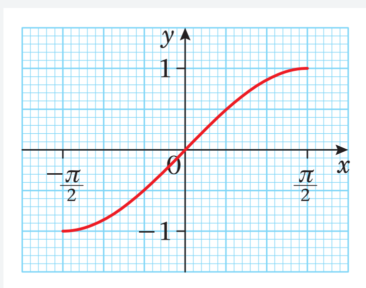
For a one-to-one function you can draw the graph of its inverse by reflecting the graph of the function in the line $y = x$. The three trigonometric functions $\sin x$, $\cos x$ and $\tan x$ only have inverse functions if their domains are restricted so that they are one-to-one functions. The notations used for these inverse functions are $\arcsin x$, $\arccos x$ and $\arctan x$ respectively ($\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$ are also used).

See Chapter 2.

Example 15

Sketch the graph of $y = \arcsin x$.

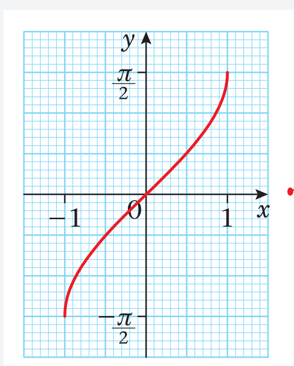
$y = \sin x,\ -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$



Step 1
Draw the graph of $y = \sin x$, with the restricted domain of $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

This is a **one-to-one** function, taking all

$y = \arcsin x$



Step 2
Reflect in the line $y = x$.

The domain of $\arcsin x$ is $-1 \leqslant x \leqslant 1$;

the range is $-\frac{\pi}{2} \leqslant \arcsin x \leqslant \frac{\pi}{2}$

Remember that the x and y coordinates of points interchange when reflecting in $y = x$. For example:

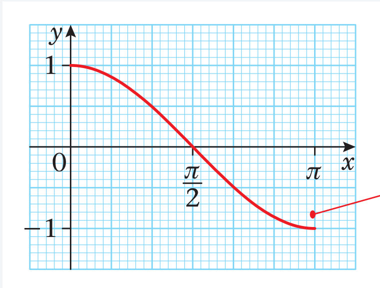
$$\left(\frac{\pi}{2}, 0\right) \rightarrow \left(0, \frac{\pi}{2}\right),\ (0, 1) \rightarrow (1, 0)$$

■ **$\arcsin x$ is the angle α, in the interval $-\frac{\pi}{2} \leqslant \alpha \leqslant \frac{\pi}{2}$, for which $\sin \alpha = x$.**

Example 16

Sketch the graph of $y = \arccos x$.

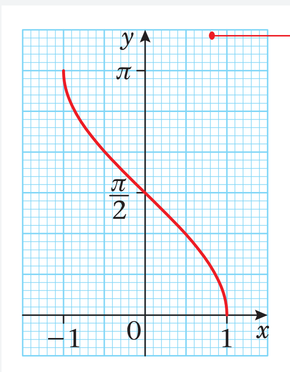
$y = \cos x,\ 0 \leqslant x \leqslant \pi.$



Step 1
Draw the graph of $y = \cos x$, with the restricted domain of $0 \leqslant x \leqslant \pi$.

This is a **one-to-one** function, taking all values in the range $\mathbf{-1 \leqslant \cos x \leqslant 1}$.

$y = \arccos x$



Step 2
Reflect in the line $y = x$.

The domain of $\arccos x$ is $-1 \leqslant x \leqslant 1$; the range is $0 \leqslant \arccos x \leqslant \pi$.

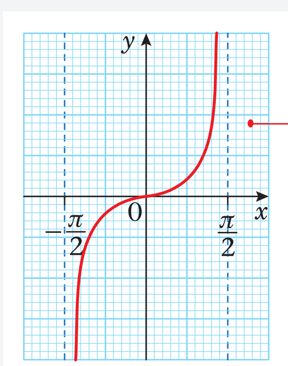
Note: $(0, 1) \to (1, 0)$, $\left(\frac{\pi}{2},\ 0\right) \to \left(0,\ \frac{\pi}{2}\right)$, $(\pi, -1) \to (-1, \pi)$.

- **arccos x is the angle α, in the interval $0 \leqslant \alpha \leqslant \pi$, for which $\cos \alpha = x$.**

Example 17

Sketch the graph of $y = \arctan x$.

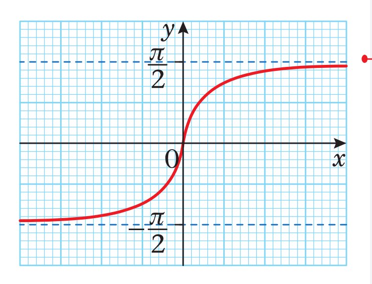
$y = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$



Step 1
Draw the graph of $y = \tan x$, with the restricted domain of $-\frac{\pi}{2} < x < \frac{\pi}{2}$

This is a **one-to-one** function, with range $\tan x \in \mathbb{R}$.

$y = \arctan x$



Step 2
Reflect in the line $y = x$.

The domain of $\arctan x$ is $x \in \mathbb{R}$; the range is $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$

■ **$\arctan x$ is the angle α, in the interval $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, for which $\tan \alpha = x$.**

Example 18

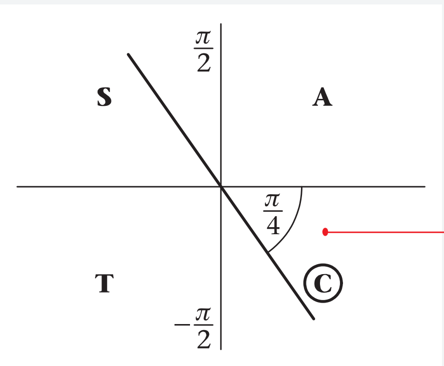
Work out, in radians, the values of

a $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$

b $\arccos(-1)$

c $\arctan(\sqrt{3})$

a

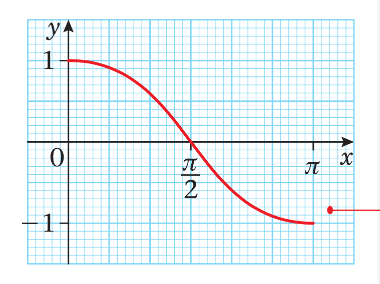


You need to solve, in the interval $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, the equation $\sin x = -\frac{\sqrt{2}}{2}$.

The angle to the horizontal is $\frac{\pi}{4}$ and, as sin is −ve, it is in the 4th quadrant.

$\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$ or -0.785 (3 s.f.)

b

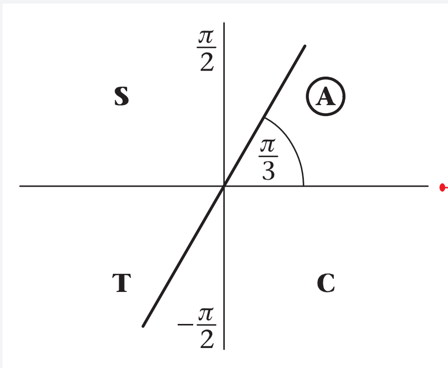


You need to solve, in the interval $0 \leqslant x \leqslant \pi$, the equation $\cos x = -1$.

Draw the graph of $y = \cos x$.

$\arccos(-1) = \pi$ or 3.14 (3 s.f.)

c



You need to solve, in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$, the equation $\tan x = \sqrt{3}$.
The angle to the horizontal is $\frac{\pi}{3}$ and, as tan is +ve, it is in the 1st quadrant.

$\arctan(\sqrt{3}) = \frac{\pi}{3}$ or 1.05 (3 s.f.)

Exercise 6E

1 Without using a calculator, work out, giving your answer in terms of π, the value of:

a $\arccos 0$ **b** $\arcsin(1)$ **c** $\arctan(-1)$ **d** $\arcsin(-\frac{1}{2})$

e $\arccos\left(-\frac{1}{\sqrt{2}}\right)$ **f** $\arctan\left(-\frac{1}{\sqrt{3}}\right)$ **g** $\arcsin\left(\sin\frac{\pi}{3}\right)$ **h** $\arcsin\left(\sin\frac{2\pi}{3}\right)$

2 Find the value of:

a $\arcsin(\frac{1}{2}) + \arcsin(-\frac{1}{2})$ **b** $\arccos(\frac{1}{2}) - \arccos(-\frac{1}{2})$ **c** $\arctan(1) - \arctan(-1)$

3 Without using a calculator, work out the values of:

a $\sin(\arcsin\frac{1}{2})$ **b** $\sin[\arcsin(-\frac{1}{2})]$

c $\tan[\arctan(-1)]$ **d** $\cos(\arccos 0)$

4 Without using a calculator, work out the exact values of:

a $\sin[\arccos(\frac{1}{2})]$ **b** $\cos[\arcsin(-\frac{1}{2})]$ **c** $\tan\left[\arccos\left(-\frac{\sqrt{2}}{2}\right)\right]$

d $\sec[\arctan(\sqrt{3})]$ **e** $\operatorname{cosec}[\arcsin(-1)]$ **f** $\sin\left[2\arcsin\left(\frac{\sqrt{2}}{2}\right)\right]$

5 Given that $\arcsin k = \alpha$, where $0 < k < 1$ and α is in radians, write down, in terms of α, the first two positive values of x satisfying the equation $\sin x = k$.

6 Given that x satisfies $\arcsin x = k$, where $0 < k < \frac{\pi}{2}$,

a state the range of possible values of x

b express, in terms of x,

i $\cos k$ **ii** $\tan k$

Given, instead, that $-\frac{\pi}{2} < k < 0$,

c how, if at all, would it affect your answers to **b**?

7 The function f is defined as $\mathrm{f}: x \to \arcsin x$, $-1 \leqslant x \leqslant 1$, and the function g is such that $\mathrm{g}(x) = \mathrm{f}(2x)$.

a Sketch the graph of $y = \mathrm{f}(x)$ and state the range of f.

b Sketch the graph of $y = \mathrm{g}(x)$.

c Define g in the form $\mathrm{g}: x \to \ldots$ and give the domain of g.

d Define g^{-1} in the form $\mathrm{g}^{-1}: x \to \ldots$

8 **a** Sketch the graph of $y = \sec x$, with the restricted domain $0 \leqslant x \leqslant \pi$, $x \neq \frac{\pi}{2}$.

b Given that $\operatorname{arcsec} x$ is the inverse function of $\sec x$, $0 \leqslant x \leqslant \pi$, $x \neq \frac{\pi}{2}$, sketch the graph of $y = \operatorname{arcsec} x$ and state the range of $\operatorname{arcsec} x$.

Mixed exercise 6F

Give any non-exact answers to equations to 1 decimal place.

1 Solve $\tan x = 2 \cot x$, in the interval $-180° \leqslant x \leqslant 90°$.

2 Given that $p = 2 \sec \theta$ and $q = 4 \cos \theta$, express p in terms of q.

3 Given that $p = \sin \theta$ and $q = 4 \cot \theta$, show that $p^2q^2 = 16(1 - p^2)$.

4 **a** Solve, in the interval $0 < \theta < 180°$,

i $\operatorname{cosec} \theta = 2 \cot \theta$ **ii** $2 \cot^2 \theta = 7 \operatorname{cosec} \theta - 8$

b Solve, in the interval $0 \leqslant \theta \leqslant 360°$,

i $\sec(2\theta - 15°) = \operatorname{cosec} 135°$ **ii** $\sec^2 \theta + \tan \theta = 3$

c Solve, in the interval $0 \leqslant x \leqslant 2\pi$,

i $\operatorname{cosec}\left(x + \frac{\pi}{15}\right) = -\sqrt{2}$ **ii** $\sec^2 x = \frac{4}{3}$

5 Given that $5 \sin x \cos y + 4 \cos x \sin y = 0$, and that $\cot x = 2$, find the value of $\cot y$.

6 Show that:

a $(\tan \theta + \cot \theta)(\sin \theta + \cos \theta) \equiv \sec \theta + \operatorname{cosec} \theta$

b $\dfrac{\operatorname{cosec} x}{\operatorname{cosec} x - \sin x} \equiv \sec^2 x$

c $(1 - \sin x)(1 + \operatorname{cosec} x) \equiv \cos x \cot x$

d $\dfrac{\cot x}{\operatorname{cosec} x - 1} - \dfrac{\cos x}{1 + \sin x} \equiv 2 \tan x$

e $\dfrac{1}{\operatorname{cosec} \theta - 1} + \dfrac{1}{\operatorname{cosec} \theta + 1} \equiv 2 \sec \theta \tan \theta$

f $\dfrac{(\sec \theta - \tan \theta)(\sec \theta + \tan \theta)}{1 + \tan^2 \theta} \equiv \cos^2 \theta$

7 **a** Show that $\dfrac{\sin x}{1+\cos x} + \dfrac{1+\cos x}{\sin x} \equiv 2\operatorname{cosec} x$.

b Hence solve, in the interval $-2\pi \leqslant x \leqslant 2\pi$, $\dfrac{\sin x}{1+\cos x} + \dfrac{1+\cos x}{\sin x} = -\dfrac{4}{\sqrt{3}}$.

8 Prove that $\dfrac{1+\cos\theta}{1-\cos\theta} \equiv (\operatorname{cosec}\theta + \cot\theta)^2$.

9 Given that $\sec A = -3$, where $\dfrac{\pi}{2} < A < \pi$,

a calculate the exact value of $\tan A$.

b Show that $\operatorname{cosec} A = \dfrac{3\sqrt{2}}{4}$.

10 Given that $\sec\theta = k$, $|k| \geqslant 1$, and that θ is obtuse, express in terms of k:

a $\cos\theta$ **b** $\tan^2\theta$ **c** $\cot\theta$ **d** $\operatorname{cosec}\theta$

11 Solve, in the interval $0 \leqslant x \leqslant 2\pi$, the equation $\sec\left(x + \dfrac{\pi}{4}\right) = 2$, giving your answers in terms of π.

12 Find, in terms of π, the value of $\arcsin(\frac{1}{2}) - \arcsin(-\frac{1}{2})$.

13 Solve, in the interval $0 \leqslant x \leqslant 2\pi$, the equation $\sec^2 x - \dfrac{2\sqrt{3}}{3}\tan x - 2 = 0$, giving your answers in terms of π.

14 **a** Factorise $\sec x \operatorname{cosec} x - 2\sec x - \operatorname{cosec} x + 2$.

b Hence solve $\sec x \operatorname{cosec} x - 2\sec x - \operatorname{cosec} x + 2 = 0$, in the interval $0 \leqslant x \leqslant 360°$.

15 Given that $\arctan(x-2) = -\dfrac{\pi}{3}$, find the value of x.

16 On the same set of axes sketch the graphs of $y = \cos x$, $0 \leqslant x \leqslant \pi$, and $y = \arccos x$, $-1 \leqslant x \leqslant 1$, showing the coordinates of points in which the curves meet the axes.

17 **a** Given that $\sec x + \tan x = -3$, use the identity $1 + \tan^2 x \equiv \sec^2 x$ to find the value of $\sec x - \tan x$.

b Deduce the value of

i $\sec x$ **ii** $\tan x$

c Hence solve, in the interval $-180° \leqslant x \leqslant 180°$, $\sec x + \tan x = -3$.

18 Given that $p = \sec\theta - \tan\theta$ and $q = \sec\theta + \tan\theta$, show that $p = \dfrac{1}{q}$.

19 **a** Prove that $\sec^4\theta - \tan^4\theta \equiv \sec^2\theta + \tan^2\theta$.

b Hence solve, in the interval $-180° \leqslant \theta \leqslant 180°$, $\sec^4\theta = \tan^4\theta + 3\tan\theta$.

20 (Although integration is not in the specification for C3, this question only requires you to know that the area under a curve can be represented by an integral.)

a Sketch the graph of $y = \sin x$ and shade in the area representing $\int_0^{\frac{\pi}{2}} \sin x \, dx$.

b Sketch the graph of $y = \arcsin x$ and shade in the area representing $\int_0^1 \arcsin x \, dx$.

c By considering the shaded areas explain why $\int_0^{\frac{\pi}{2}} \sin x \, dx + \int_0^1 \arcsin x \, dx = \dfrac{\pi}{2}$.

Summary of key points

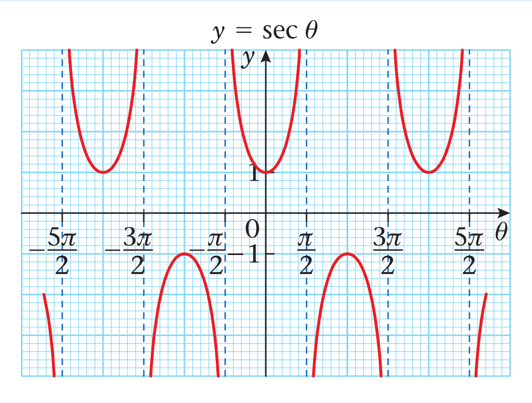
1 • $\sec\theta = \dfrac{1}{\cos\theta}$ {sec θ is undefined when $\cos\theta = 0$, i.e. at $\theta = (2n+1)\,90°$, $n \in \mathbb{Z}$}

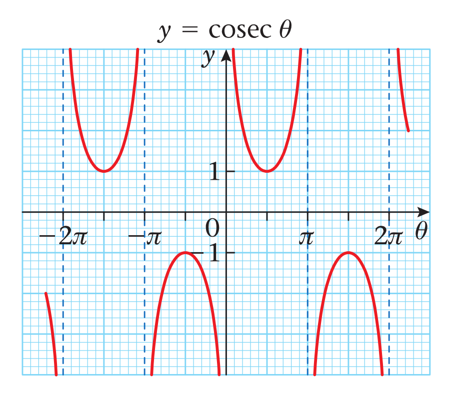
• $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ {cosec θ is undefined when $\sin\theta = 0$, i.e. at $\theta = 180n°$, $n \in \mathbb{Z}$}

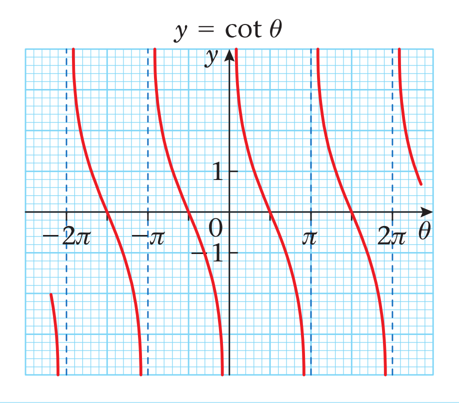
• $\cot\theta = \dfrac{1}{\tan\theta}$ {cot θ is undefined when $\tan\theta = 0$, i.e. at $\theta = 180n°$, $n \in \mathbb{Z}$}

• $\cot\theta$ can also be written as $\dfrac{\cos\theta}{\sin\theta}$.

2 The graphs of sec θ, cosec θ and cot θ are



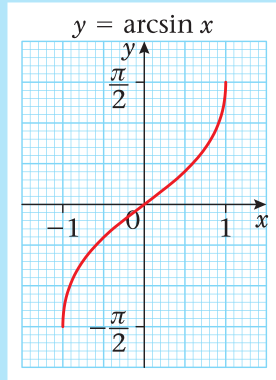




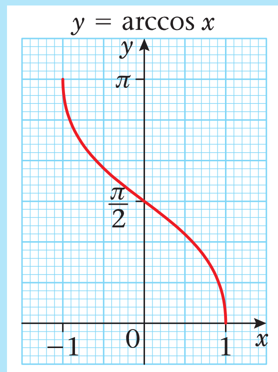
3 Two further Pythagorean identities, derived from $\sin^2\theta + \cos^2\theta \equiv 1$, are

$$1 + \tan^2\theta \equiv \sec^2\theta \quad \text{and} \quad 1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$$

4 The inverse function of $\sin x$, $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, is called $\arcsin x$; it has domain $-1 \leqslant x \leqslant 1$ and range $-\frac{\pi}{2} \leqslant \arcsin x \leqslant \frac{\pi}{2}$



5 The inverse function of $\cos x$, $0 \leqslant x \leqslant \pi$, is called $\arccos x$; it has domain $-1 \leqslant x \leqslant 1$ and range $0 \leqslant \arccos x \leqslant \pi$.



6 The inverse function of $\tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, is called $\arctan x$; it has domain $x \in \mathbb{R}$ and range $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$.