2 Functions

After completing this chapter you should be able to

1 represent a mapping by a diagram, by an equation and by a graph

2 understand the terms function, domain and range

3 combine two or more functions to make a composite function

4 know the difference between a ‘one to one’ and ‘many to one’ function

5 know how to find the inverse of a function

6 know the relationship between the graph of a function and its inverse.

The electricity bill for this house would be high!

There are many examples of functions in real life. One interesting case involves electricity charges where you are charged different amounts per unit dependent upon how many units you use. This is a typical case:

‘The charge is 10p per kW hour up to and including a usage of 250 kW hours and 8p per kW hour after that’.

2.1 **A mapping transforms one set of numbers into a different set of numbers. The mapping can be described in words or through an algebraic equation. It can also be represented by a Cartesian graph.**

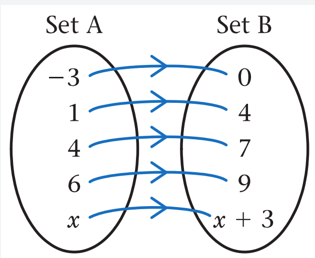
Example 1

Draw mapping diagrams and graphs for the following operations:

a 'add 3' on the set $\{-3, 1, 4, 6, x\}$ **b** 'square' on the set $\{-1, 1, -2, 2, x\}$

a Operation
Add 3

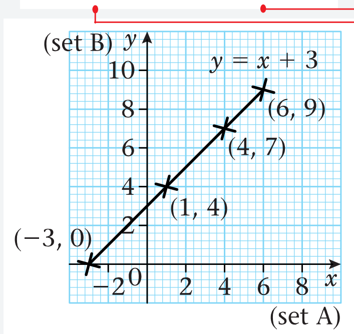
Mapping diagram



This set is called the **range**.

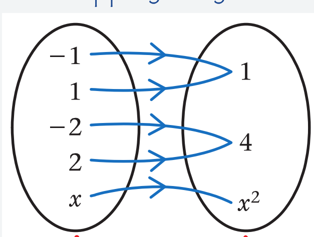
This set is called the **domain**.

Graph



b Operation
Square

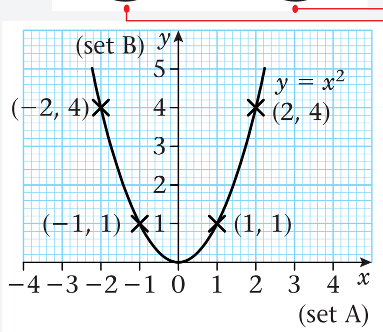
Mapping diagram



The range.

The domain.

Graph



Exercise 2A

1 Draw mapping diagrams and graphs for the following operations:

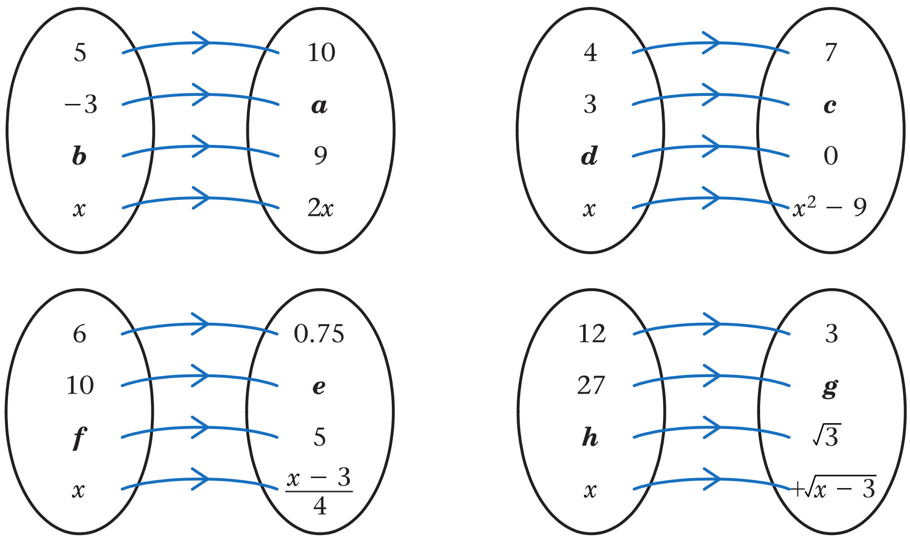
a ‘subtract 5’ on the set $\{10, 5, 0, -5, x\}$

b ‘double and add 3’ on the set $\{-2, 2, 4, 6, x\}$

c ‘square and then subtract 1’ on the set $\{-3, -1, 0, 1, 3, x\}$

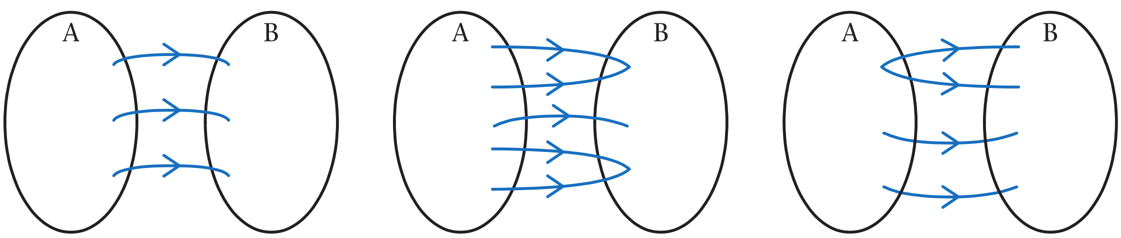
d ‘the positive square root’ on the set $\{-4, 0, 1, 4, 9, x\}$.

2 Find the missing numbers $\boldsymbol{a}$ to $\boldsymbol{h}$ in the following mapping diagrams:



2.2 A function is a special mapping such that every element of set A (the domain) is mapped to exactly one element of set B (the range).

■ **A good way to remember this is**



one-to-one function | many-to-one function | not a function

You can write functions in two different ways:

$$f(x) = 2x + 1 \quad \text{OR} \quad f : x \rightarrow 2x + 1$$

This is the function ‘double and add 1’.

Example 2

Given that the function $g(x) = 2x^2 + 3$, find:

a the value of $g(2)$

b the value of a such that $g(a) = 35$

c the range of the function.

a $g(2) = 2(2)^2 + 3 = 11$

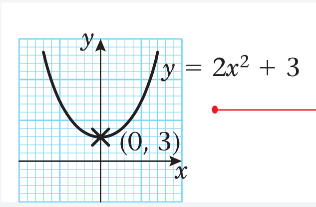
b $g(a) = 2a^2 + 3 = 35$

$2a^2 = 32$

$a^2 = 16$

$a = \pm 4$

c



Range of $g(x)$ is $g(x) \geqslant 3$

Substitute $x = 2$ in the formula.

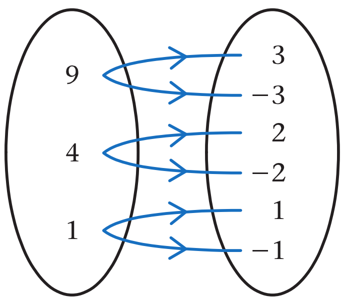
Substitute $x = a$ and $g(a) = 35$.

Sketch the function $g(x)$. The range is the values that y takes.

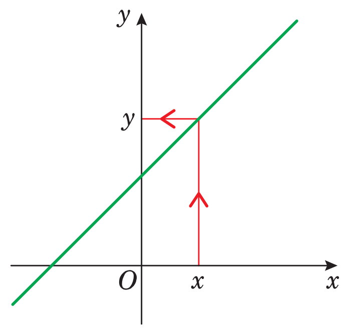
Example 3

Which of the following mappings represent functions? Give reasons for your answers.

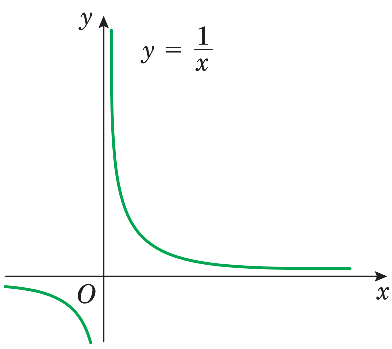
a



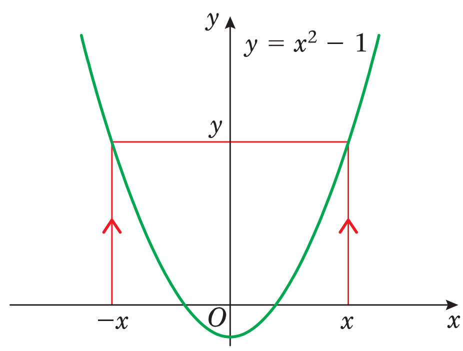
b



c



d



a Every element of set A gets mapped to two elements in set B. Therefore this is *not* a function. It is an operation (square root).

b Every value of x gets mapped to one value of y. This is therefore a function. It can be written in the form $y = ax + b$ for some constants a and b. This type of function is called

one-to-one because every element of the range comes exactly from one element in the domain.

c On the sketch of $y = \frac{1}{x}$ you can see that $x = 0$ does not get mapped anywhere. Therefore not all elements of set A get mapped to elements in set B. Hence it is *not* a function.

d This is a function. It is called a **many-to-one** function because two elements of the domain get mapped to one element in the range.

Exercise 2B

1 Find:

a f(3) where $f(x) = 5x + 1$ **b** g(−2) where $g(x) = 3x^2 - 2$

c h(0) where $h: x \to 3^x$ **d** j(−2) where $j: x \to 2^{-x}$

2 Calculate the value(s) of a, b, c and d given that:

a $p(a) = 16$ where $p(x) = 3x - 2$ **b** $q(b) = 17$ where $q(x) = x^2 - 3$

c $r(c) = 34$ where $r(x) = 2(2^x) + 2$ **d** $s(d) = 0$ where $s(x) = x^2 + x - 6$

3 For the following functions

i sketch the graph of the function

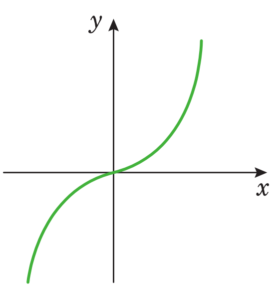
ii state the range

iii describe if the function is one-to-one or many-to-one.

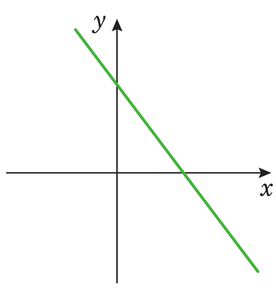
a $m(x) = 3x + 2$ **b** $n(x) = x^2 + 5$ **c** $p(x) = \sin(x)$ **d** $q(x) = x^3$

4 State whether or not the following graphs represent functions. Give reasons for your answers and describe the type of function.

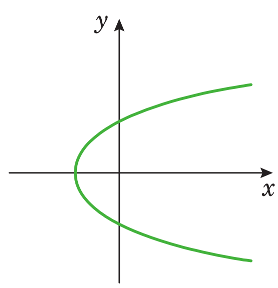
a



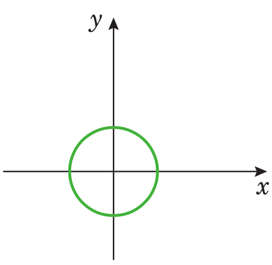
b



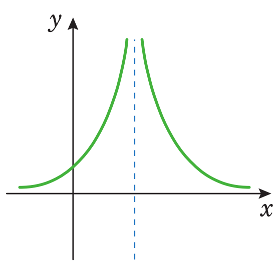
c



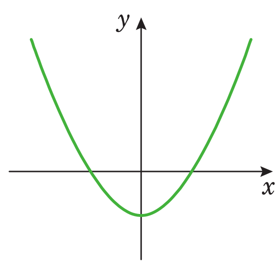
d



e

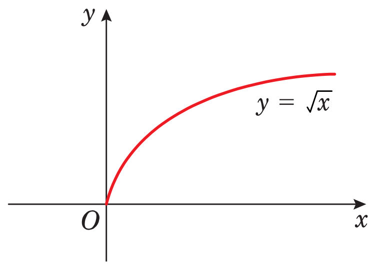


f



2.3 Many mappings can be made into functions by changing the domain.

Consider $y = \sqrt{x}$



If the domain is all of the real numbers $\{x \in \mathbb{R}\}$, then this is not a function because values of x less than 0 do not get mapped anywhere.

If you restrict the domain to $x \geqslant 0$, all of set A gets mapped to exactly one element of set B.

We can write this function $f(x) = \sqrt{x}$, with domain $\{x \in \mathbb{R}, x \geqslant 0\}$.

Example 4

Find the range of the following functions:

a $f(x) = 3x - 2$, domain $\{x = 1, 2, 3, 4\}$

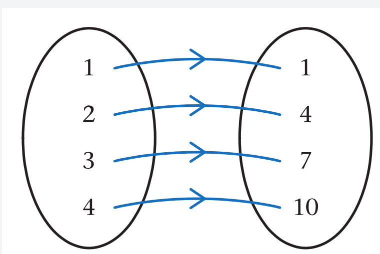
b $g(x) = x^2$, domain $\{x \in \mathbb{R}, -5 \leqslant x \leqslant 5\}$

c $h(x) = \dfrac{1}{x}$ domain $\{x \in \mathbb{R}, 0 < x \leqslant 3\}$

State if the functions are one-to-one or many-to-one.

a $f(x) = 3x - 2, \{x = 1, 2, 3, 4\}$

Here the domain is discrete as it only has integer values. Draw a mapping diagram.

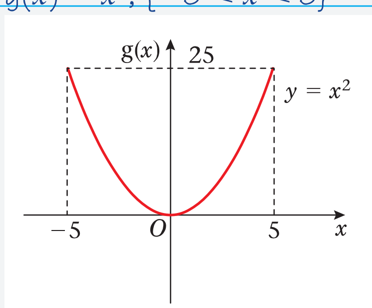


Range of $f(x)$ is $\{1, 4, 7, 10\}$.

$f(x)$ is one-to-one.

b $g(x) = x^2, \{-5 \leqslant x \leqslant 5\}$

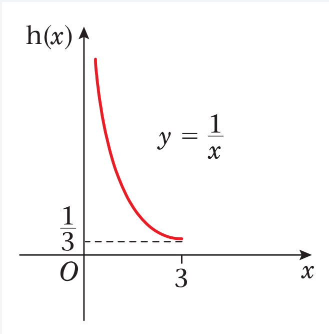
Here the domain is continuous. It takes all values between −5 and 5. Sketch a graph.



Range of $g(x)$ is $0 \leqslant g(x) \leqslant 25$.

$g(x)$ is many-to-one.

c $\mathrm{h}(x) = \dfrac{1}{x}\ \{x \in \mathbb{R}, 0 < x \leqslant 3\}$



Sketch a graph.

Range of $\mathrm{h}(x)$ is $\mathrm{h}(x) \geqslant \frac{1}{3}$.

$\mathrm{h}(x)$ is one-to-one.

Example 5

The function $\mathrm{f}(x)$ is defined by

$$\mathrm{f}(x) = \begin{cases} 5 - 2x & x < 1 \\ x^2 + 3 & x \geqslant 1 \end{cases}$$

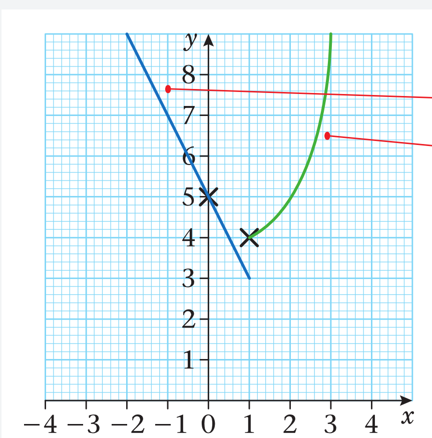
Note: This function consists of two parts – one linear (for $x < 1$), the other quadratic (for $x \geqslant 1$). A useful tip in drawing the function is to sketch both parts separately and also to find the value of both parts at $x = 1$.

a Sketch $\mathrm{f}(x)$ stating its range.

b Find the values of a such that $\mathrm{f}(a) = 19$.

a $\mathrm{f}(1)$, using $\mathrm{f}(x) = 5 - 2x = 5 - 2 = 3$

$\mathrm{f}(1)$, using $\mathrm{f}(x) = x^2 + 3 = 1 + 3 = 4$

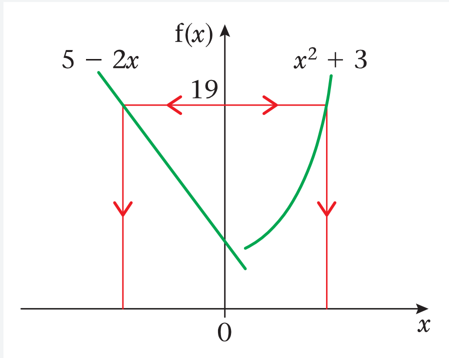


For $x < 1$, $\mathrm{f}(x)$ is linear. It has gradient -2 and passes through 5 on the y axis.
For $x > 1$, $\mathrm{f}(x)$ is a $\cup$-shaped quadratic. You need to calculate the value of $\mathrm{f}(1)$ on both curves.

The range is the values that y takes and therefore $\mathrm{f}(x) > 3$.

Note that $\mathrm{f}(x) \neq 3$ at $x = 1$
so $\quad \mathrm{f}(x) > 3$
not $\quad \mathrm{f}(x) \geqslant 3$

b



The positive point is where

$x^2 + 3 = 19$

$x^2 = 16$

$x = \pm 4$

$x = 4$

The negative point is where

$5 - 2x = 19$

$-2x = 14$

$x = -7$

The two values of a are 4 and -7.

For many questions on functions it becomes easier to sketch a graph. There are 2 values such that $f(a) = 19$.

Ignore -4 as $x > 1$.

Use both expressions as there are two distinct points.

Exercise 2C

1 The functions below are defined for the discrete domains.

i Represent each function on a mapping diagram, writing down the elements in the range.

ii State if the function is one-to-one or many-to-one.

a $f(x) = 2x + 1$ for the domain $\{x = 1, 2, 3, 4, 5\}$.

b $g(x) = +\sqrt{x}$ for the domain $\{x = 1, 4, 9, 16, 25, 36\}$.

c $h(x) = x^2$ for the domain $\{x = -2, -1, 0, 1, 2\}$.

d $j(x) = \dfrac{2}{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$.

2 The functions below are defined for continuous domains.

i Represent each function on a graph.

ii State the range of the function.

iii State if the function is one-to-one or many-to-one.

a $m(x) = 3x + 2$ for the domain $\{x > 0\}$.

b $n(x) = x^2 + 5$ for the domain $\{x \geqslant 2\}$.

c $p(x) = 2\sin x$ for the domain $\{0 \leqslant x \leqslant 180\}$.

d $q(x) = +\sqrt{x + 2}$ for the domain $\{x \geqslant -2\}$.

3 The following mappings f and g are defined on all the real numbers by

$$f(x) = \begin{cases} 4 - x & x < 4 \\ x^2 + 9 & x \geqslant 4 \end{cases} \qquad g(x) = \begin{cases} 4 - x & x < 4 \\ x^2 + 9 & x > 4 \end{cases}$$

Explain why f(x) is a function and g(x) is not.
Sketch the function f(x) and find

a f(3)

b f(10)

c the value(s) of a such that f(a) = 90.

4 The function s is defined by

$$s(x) = \begin{cases} x^2 - 6 & x < 0 \\ 10 - x & x \geqslant 0 \end{cases}$$

a Sketch s(x).

b Find the value(s) of a such that s(a) = 43.

c Find the values of the domain that get mapped to themselves in the range.

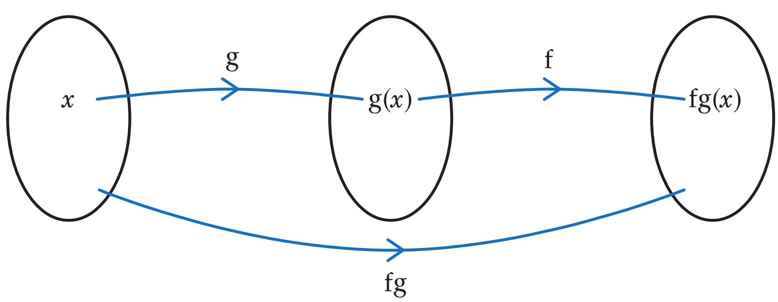
5 The function g is defined by g(x) = $cx + d$ where c and d are constants to be found. Given g(3) = 10 and g(8) = 12 find the values of c and d.

6 The function f is defined by f(x) = $ax^3 + bx - 5$ where a and b are constants to be found. Given that f(1) = −4 and f(2) = 9, find the values of the constants a and b.

7 The function h is defined by h(x) = $x^2 - 6x + 20$ $\{x \geqslant a\}$. Given that h(x) is a one-to-one function find the smallest possible value of the constant a.

Hint: Complete the square for h(x).

2.4 You can combine two or more basic functions to make a new more complex function.



■ **fg(x) means apply g first, followed by f. fg(x) is called a composite function.**

Example 6

Given f(x) = x^2 and g(x) = $x + 1$, find:
a fg(1) **b** fg(3) **c** fg(x)

a $fg(1) = f(1+1)$	$g(1) = 1 + 1$
$= 2^2$	$f(2) = 2^2$
$= 4$	
b $fg(3) = f(3+1)$	$g(3) = 3 + 1$
$= 4^2$	$f(4) = 4^2$
$= 16$	
c $fg(x) = f(x+1)$	$g(x) = x + 1$
$= (x+1)^2$	$f(x + 1) = (x + 1)^2$

Example 7

The functions f and g are defined by $f(x) = 3x + 2$ and $g(x) = x^2 + 4$. Find:

a the function $fg(x)$

b the function $gf(x)$

c the function $f^2(x)$

d the values of b such that $fg(b) = 62$

Note: $f^2(x)$ is $ff(x)$.

a $fg(x) = f(x^2 + 4)$	g acts on x first, mapping it to $x^2 + 4$.
$= 3(x^2 + 4) + 2$	f acts on the result. f 'trebles and then adds 2'.
$= 3x^2 + 14$	Simplify answer.
b $gf(x) = g(3x + 2)$	f acts on x first, mapping it to $3x + 2$.
$= (3x + 2)^2 + 4$	g acts on the result. g 'squares and then adds 4'.
$= 9x^2 + 12x + 8$	Simplify answer.
c $f^2(x) = f(3x + 2)$	f maps x to $3x + 2$.
$= 3(3x + 2) + 2$	f acts on the result. f 'trebles numbers then adds 2'.
$= 9x + 8$	
d $fg(x) = 3x^2 + 14$	
If $fg(b) = 62$	
$3b^2 + 14 = 62$	Set up and solve an equation in b.
$3b^2 = 48$	
$b^2 = 16$	
$b = \pm 4$	

Example 8

The functions m(x), n(x) and p(x) are defined by $m(x) = \frac{1}{x}$, $n(x) = 2x + 4$, $p(x) = x^2 - 2$.
Find in terms of m, n and p the functions

a $\frac{2}{x} + 4$ **b** $4x^2 + 16x + 14$ **c** $\frac{1}{4x + 12}$

a $\frac{2}{x} + 4 = 2\left(\frac{1}{x}\right) + 4$
$= 2m(x) + 4$
$= nm(x)$

$m(x) = \frac{1}{x}$
$n(x) = 2x + 4$

b $4x^2 + 16x + 14 = (2x + 4)^2 - 2$
$= [n(x)]^2 - 2$
$= pn(x)$

$n(x) = 2x + 4$
$p(x) = x^2 - 2$

c $\frac{1}{4x + 12} = \frac{1}{2(2x + 4) + 4}$
$= \frac{1}{2n(x) + 4}$
$= \frac{1}{nn(x)}$
$= mnn(x)$
$= mn^2(x)$

$n(x) = 2x + 4$
$n(x) = 2x + 4$
$m(x) = \frac{1}{x}$
$nn(x) = n^2(x)$

Exercise 2D

1 Given the functions $f(x) = 4x + 1$, $g(x) = x^2 - 4$ and $h(x) = \frac{1}{x}$, find expressions for the functions:

a fg(x) **b** gf(x) **c** gh(x) **d** fh(x) **e** $f^2(x)$

2 For the following functions f(x) and g(x), find the composite functions fg(x) and gf(x). In each case find a suitable domain and the corresponding range when

a $f(x) = x - 1$, $g(x) = x^2$ **b** $f(x) = x - 3$, $g(x) = +\sqrt{x}$ **c** $f(x) = 2^x$, $g(x) = x + 3$

3 If $f(x) = 3x - 2$ and $g(x) = x^2$, find the number(s) a such that fg(a) = gf(a).

4 Given that $s(x) = \frac{1}{x - 2}$ and $t(x) = 3x + 4$ find the number m such that ts(m) = 16.

5 The functions l(x), m(x), n(x) and p(x) are defined by $l(x) = 2x + 1$, $m(x) = x^2 - 1$, $n(x) = \frac{1}{x + 5}$ and $p(x) = x^3$. Find in terms of l, m, n and p the functions:

a $4x + 3$ **b** $4x^2 + 4x$ **c** $\frac{1}{x^2 + 4}$ **d** $\frac{2}{x + 5} + 1$

e $(x^2 - 1)^3$ **f** $2x^2 - 1$ **g** x^{27}

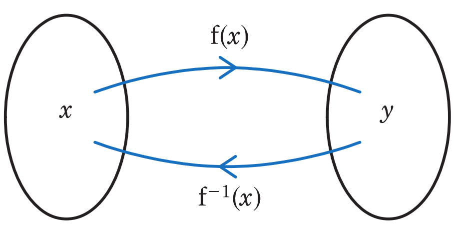
6 If $m(x) = 2x + 3$ and $n(x) = \dfrac{x - 3}{2}$, prove that $mn(x) = x$.

7 If $s(x) = \dfrac{3}{x + 1}$ and $t(x) = \dfrac{3 - x}{x}$, prove that $st(x) = x$.

8 If $f(x) = \dfrac{1}{x + 1}$, prove that $f^2(x) = \dfrac{x + 1}{x + 2}$. Hence find an expression for $f^3(x)$.

2.5 The inverse function performs the opposite operation to the function. It takes elements of the range and maps them back into elements of the domain. For this reason inverse functions only exist for one-to-one functions.

The inverse of $f(x)$ is written $f^{-1}(x)$



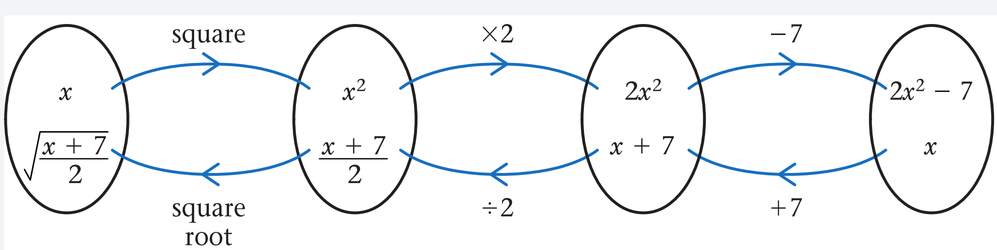
$ff^{-1}(x) = f^{-1}f(x) = x$

Function		Inverse	
$f(x) = x + 4$	'add 4'	'subtract 4'	$f^{-1}(x) = x - 4$
$g(x) = 5x$	'times 5'	'divide by 5'	$g^{-1}(x) = \dfrac{x}{5}$
$h(x) = 4x + 2$	'times 4, add 2'	'subtract 2, divide by 4'	$h^{-1}(x) = \dfrac{x - 2}{4}$

For many straightforward functions the inverse can be found using a flow chart.

Example 9

Find the inverse of the function $h(x) = 2x^2 - 7$.



Draw a flow chart for the function.

Therefore,

$$h^{-1}(x) = \sqrt{\frac{x + 7}{2}}$$

Example 10

Find the inverse of the function $f(x) = \dfrac{3}{x-1}$, $\{x \in \mathbb{R}, x \neq 1\}$, by changing the subject of the formula.

Let $y = f(x)$

$$
\begin{aligned}
y &= \frac{3}{x-1} && \text{(cross multiply)} \\
y(x-1) &= 3 && \text{(remove bracket)} \\
yx - y &= 3 && \text{(add } y\text{)} \\
yx &= 3 + y && \text{(divide by } y\text{)} \\
x &= \frac{3+y}{y}
\end{aligned}
$$

Rearrange to make x the subject of the formula.

Therefore $\quad f^{-1}(x) = \dfrac{3+x}{x}$

Define $f^{-1}(x)$ in terms of x.

$$
\begin{aligned}
f(4) &= \frac{3}{4-1} = \frac{3}{3} = 1 \\
f^{-1}(1) &= \frac{3+1}{1} = \frac{4}{1} = 4
\end{aligned}
$$

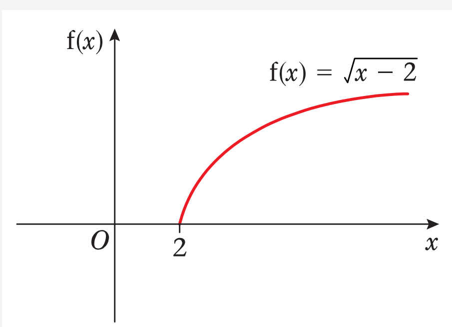
$$
\begin{tikzcd}
4 \arrow[r, bend left, "f(x)"] & 1 \arrow[l, bend left, "f^{-1}(x)"]
\end{tikzcd}
$$

Check to see that at least one element works. Try 4.
Note that $f^{-1}f(4) = 4$.

Example 11

The function $f(x)$ is defined by $f(x) = \sqrt{x-2}$ $\{x \in \mathbb{R}, x \geqslant 2\}$. Find the function $f^{-1}(x)$ in a similar form stating its domain.

The range of the function is the domain of the inverse function and vice versa. This fact can be used to solve many problems where the domain is not all of $\mathbb{R}$.



Sketch the function for the values of x given.

The range of the function is $f(x) \geqslant 0$. Therefore the domain of the inverse function is $x \geqslant 0$.

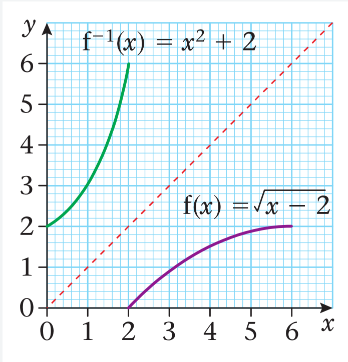
The range of the function = the domain of the inverse function.

$$
\begin{aligned}
y &= \sqrt{x-2} \\
y^2 &= x - 2 \\
x &= y^2 + 2
\end{aligned}
$$

Change the subject of the formula.

The inverse function is $f^{-1}(x) = x^2 + 2$ $\{x \in \mathbb{R}, x \geqslant 0\}$

Always write your function in terms of x.



Note that the graph of $f^{-1}(x)$ is a reflection of $f(x)$ in the line $y = x$. This is because the reflection transforms y to x and x to y.

Example 12

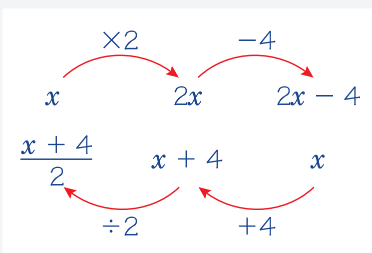
If $g(x)$ is defined as $g(x) = 2x - 4$ $\{x \in \mathbb{R}, x \geqslant 0\}$

a Calculate $g^{-1}(x)$.

b Sketch the graphs of both functions on the same set of axes.

c What is the connection between the graphs?

a

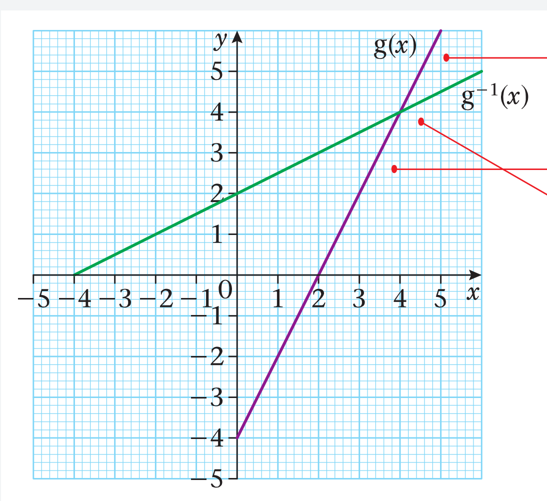


Use a flow diagram.

$$g^{-1}(x) = \frac{x+4}{2} \quad \{x \geqslant -4\}$$

The domain of the inverse function is the same as the range of the function. (See graph below.)

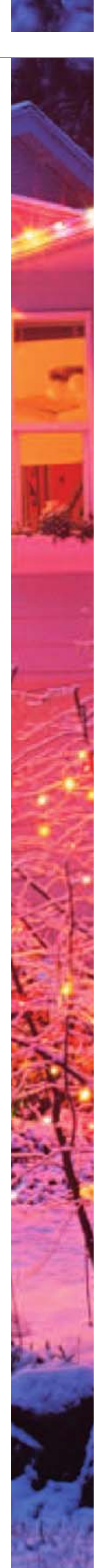
b



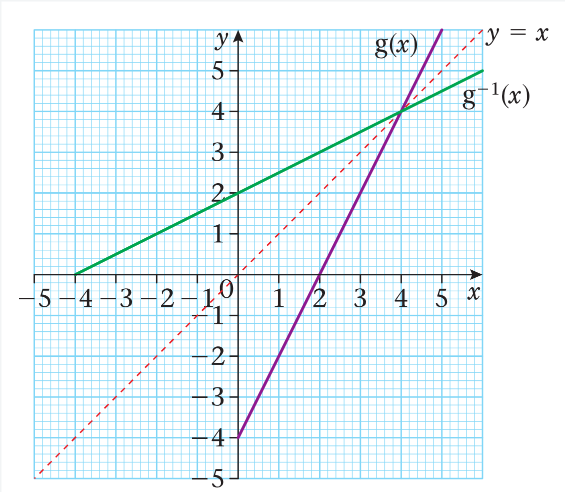
The range of the function is $g(x) \geqslant -4$, so this is the domain of the inverse function.

$g(x)$ is a linear function with gradient 2 passing through the y axis at -4. Its domain is $x \geqslant 0$.

$g^{-1}(x)$ is also a linear function with gradient 0.5 passing through the y axis at 2. Its domain is $x \geqslant -4$.



c The graphs of $g^{-1}(x)$ and $g(x)$ are mirror images of each other in the line $y = x$.



Example 13

The function $f(x)$ is defined by $f(x) = x^2 - 3$ $\{x \in \mathbb{R}, x > 0\}$.

a Find $f^{-1}(x)$ in similar terms.

b Sketch $f^{-1}(x)$.

c Find values of x such that $f(x) = f^{-1}(x)$.

a Let $y = f(x)$

$y = x^2 - 3$

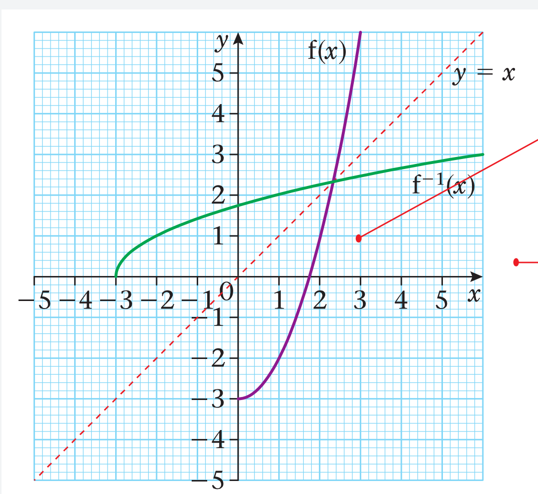
$y + 3 = x^2$

$x = \sqrt{y + 3}$

$f^{-1}(x) = \sqrt{x + 3}$ $\{x \in \mathbb{R}, x \geqslant -3\}$

Change subject of formula.

b



$f(x)$ is a $\bigcup$-shaped quadratic with minimum point $(0, -3)$. The range of the function is $f(x) \geqslant -3$.
The domain of the inverse function has the same values.

First sketch $f(x)$.
Then reflect $f(x)$ in the line $y = x$.

c When $f(x) = f^{-1}(x)$

$f(x) = x$

$x^2 - 3 = x$

$x^2 - x - 3 = 0$

So $x = \dfrac{1 + \sqrt{13}}{2}$

This is easier than solving $\sqrt{x+3} = x^2 - 3$.

Solve using $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$f(x)$ and $f^{-1}(x)$ meet where x and y are both positive.

Exercise 2E

1 For the following functions $f(x)$, sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same set of axes. Determine also the equation of $f^{-1}(x)$.

a $f(x) = 2x + 3\ \{x \in \mathbb{R}\}$

b $f(x) = \dfrac{x}{2}\ \{x \in \mathbb{R}\}$

c $f(x) = \dfrac{1}{x}\ \{x \in \mathbb{R}, x \neq 0\}$

d $f(x) = 4 - x\ \{x \in \mathbb{R}\}$

e $f(x) = x^2 + 2\ \{x \in \mathbb{R}, x \geqslant 0\}$

f $f(x) = x^3\ \{x \in \mathbb{R}\}$

2 Determine which of the functions in Question 1 are self inverses. (That is to say the function and its inverse are identical.)

3 Explain why the function $g(x) = 4 - x\ \{x \in \mathbb{R}, x > 0\}$ is not identical to its inverse.

4 For the following functions $g(x)$, sketch the graphs of $g(x)$ and $g^{-1}(x)$ on the same set of axes. Determine the equation of $g^{-1}(x)$, taking care with its domain.

a $g(x) = \dfrac{1}{x}\ \{x \in \mathbb{R}, x \geqslant 3\}$

b $g(x) = 2x - 1\ \{x \in \mathbb{R}, x \geqslant 0\}$

c $g(x) = \dfrac{3}{x - 2}\ \{x \in \mathbb{R}, x > 2\}$

d $g(x) = \sqrt{x - 3}\ \{x \in \mathbb{R}, x \geqslant 7\}$

e $g(x) = x^2 + 2\ \{x \in \mathbb{R}, x > 4\}$

f $g(x) = x^3 - 8\ \{x \in \mathbb{R}, x \leqslant 2\}$

5 The function $m(x)$ is defined by $m(x) = x^2 + 4x + 9$ $\{x \in \mathbb{R}, x > a\}$ for some constant a. If $m^{-1}(x)$ exists, state the least value of a and hence determine the equation of $m^{-1}(x)$. State its domain.

Hint: Completing the square helps in these types of questions.

6 Determine $t^{-1}(x)$ if the function $t(x)$ is defined by $t(x) = x^2 - 6x + 5\ \{x \in \mathbb{R}, x \geqslant 5\}$.

7 The function $h(x)$ is defined by $h(x) = \dfrac{2x + 1}{x - 2}\ \{x \in \mathbb{R}, x \neq 2\}$.

a What happens to the function as x approaches 2?

b Find $h^{-1}(3)$.

c Find $h^{-1}(x)$, stating clearly its domain.

d Find the elements of the domain that get mapped to themselves by the function.

8 The function f(x) is defined by $f(x) = 2x^2 - 3$ $\{x \in \mathbb{R}, x < 0\}$. Determine

a $f^{-1}(x)$ clearly stating its domain

b the values of a for which $f(a) = f^{-1}(a)$.

Mixed exercise 2F

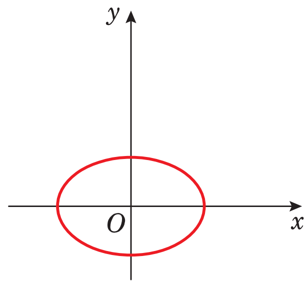
1 Categorise the following as

i not a function

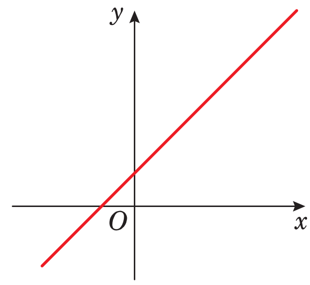
ii a one-to-one function

iii a many-to-one function.

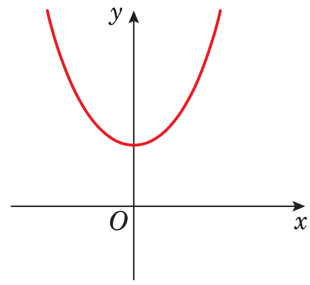
a



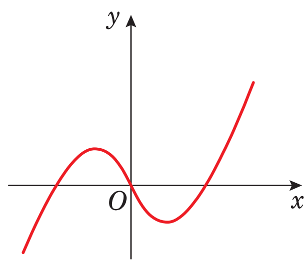
b



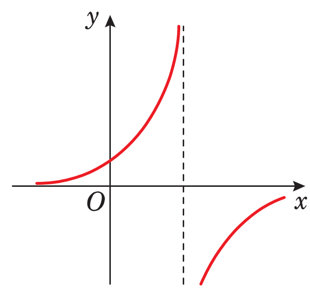
c



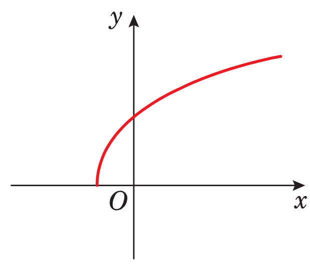
d



e



f



2 The following functions f(x), g(x) and h(x) are defined by

$$f(x) = 4(x - 2) \quad \{x \in \mathbb{R}, x \geqslant 0\}$$
$$g(x) = x^3 + 1 \quad \{x \in \mathbb{R}\}$$
$$h(x) = 3^x \quad \{x \in \mathbb{R}\}$$

a Find f(7), g(3) and h(−2).

b Find the range of f(x) and the range of g(x).

c Find $g^{-1}(x)$.

d Find the composite function fg(x).

e Solve gh(a) = 244.

3 The function n(x) is defined by

$$n(x) = \begin{cases} 5 - x & x \leqslant 0 \\ x^2 & x > 0 \end{cases}$$

a Find n(−3) and n(3).

b Find the value(s) of a such that n(a) = 50.

4 The function g(x) is defined as $g(x) = 2x + 7$ $\{x \in \mathbb{R}, x \geqslant 0\}$.

a Sketch g(x) and find the range.

b Determine $g^{-1}(x)$, stating its domain.

c Sketch $g^{-1}(x)$ on the same axes as g(x), stating the relationship between the two graphs.

5 The functions f and g are defined by

$$f : x \to 4x - 1 \quad \{x \in \mathbb{R}\}$$
$$g : x \to \frac{3}{2x - 1} \quad \{x \in \mathbb{R}, x \neq \tfrac{1}{2}\}$$

Find in its simplest form:

a the inverse function f^{-1}

b the composite function gf, stating its domain

c the values of x for which $2f(x) = g(x)$, giving your answers to 3 decimal places. *E*

6 The function f(x) is defined by

$$f(x) = \begin{cases} -x & x \leqslant 1 \\ x - 2 & x > 1 \end{cases}$$

a Sketch the graph of $f(x)$ for $-2 \leqslant x \leqslant 6$.

b Find the values of x for which $f(x) = -\frac{1}{2}$. *E*

7 The function f is defined by

$$f : x \to \frac{2x + 3}{x - 1} \quad \{x \in \mathbb{R}, x > 1\}$$

a Find $f^{-1}(x)$.

b Find **i** the range of $f^{-1}(x)$

ii the domain of $f^{-1}(x)$. *E*

8 The functions f and g are defined by

$$f : x \to \frac{x}{x - 2} \quad \{x \in \mathbb{R}, x \neq 2\}$$
$$g : x \to \frac{3}{x} \quad \{x \in \mathbb{R}, x \neq 0\}$$

a Find an expression for $f^{-1}(x)$.

b Write down the range of $f^{-1}(x)$.

c Calculate gf(1.5).

d Use algebra to find the values of x for which $g(x) = f(x) + 4$. *E*

9 The functions f and g are given by

$$f : x \to \frac{x}{x^2 - 1} - \frac{1}{x + 1} \quad \{x \in \mathbb{R}, x > 1\}$$
$$g : x \to \frac{2}{x} \quad \{x \in \mathbb{R}, x > 0\}$$

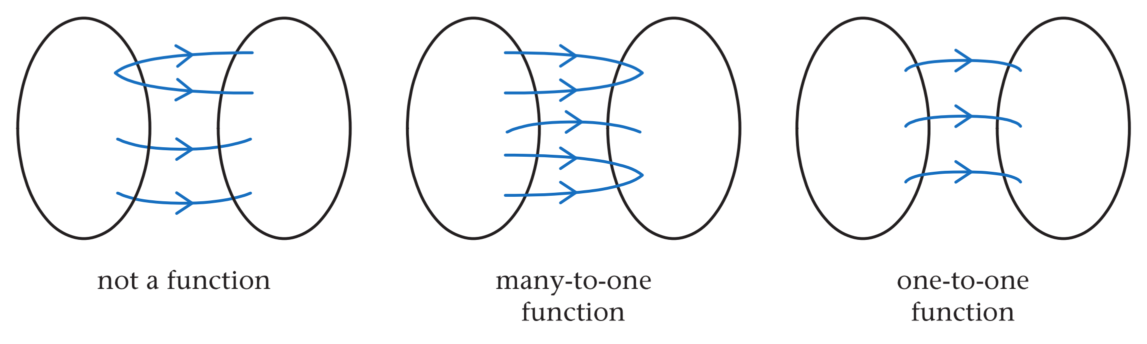
a Show that $f(x) = \dfrac{1}{(x - 1)(x + 1)}$.

b Find the range of $f(x)$.

c Solve $gf(x) = 70$. *E*

Summary of key points

1 A function is a special mapping such that every element of the domain is mapped to exactly one element in the range.



2 A one-to-one function is a special function where every element of the range has been mapped from exactly one element of the domain.

3 Many mappings can be made into functions by changing the domain. For example, the mapping 'positive square root' can be changed into the function $f(x) = \sqrt{x}$ by having a domain of $x \geqslant 0$.

4 If we combine two or more functions we can create a composite function. The function below is written $fg(x)$ as g acts on x first, then f acts on the result. For example,

$g(x) = 2x + 3$, $f(x) = x^2$
$fg(4) = f(2 \times 4 + 3) = f(11) = 11^2 = 121$

Similarly

$fg(x) = (2x + 3)^2$

5 The inverse of a function $f(x)$ is written $f^{-1}(x)$ and performs the opposite operation(s) to the function. To calculate the inverse function you change the subject of the formula. For example, the inverse function of $g(x) = 4x - 3$ is

$$g^{-1}(x) = \frac{x + 3}{4}$$

6 The range of the function is the domain of the inverse function and vice versa.

7 The graph of $f^{-1}(x)$ is a reflection of $f(x)$ in the line $y = x$.